A Nonlinear Model for Financial Dynamics

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Abstract

We propose a new nonlinear model for describing the dynamics of financial markets. The basic concept of this model is based on the microstructure model proposed by Bouchaud and Cont (1998). However, we consider both excess demand and liquidity as unobserved state variables. The resulting partially observed system is estimated by utilizing the local linear approximation method proposed by Iino and Ozaki (2000a).

Keywords: nonlinear stochastic differential equations, state space models, conditional moments, local linearization, finance, market microstructure

1 Introduction

An extraordinary number of models have been proposed in the financial literature to describe the dynamics of financial markets. Although most of the proposed models concentrate on the dynamics of conditional variance, exceptionally, some physicists have proposed phenomenological models based on identification of the different processes that influence demand and supply. One of the more remarkable models, proposed by Bouchaud and Cont (1998), is defined by

\[ dP_t = \lambda \phi_t, \]

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where $P_t$ is asset price, $\lambda$ is (the inverse of) market liquidity, and $\phi$ is excess demand. The basic concept of their model is based on the market microstructure theory (O’Hara, 1995), in which the price $P_t$ is driven by the excess demand $\phi_t$, and the amplitude of price changes is dependent on the liquidity $1/\lambda$ of the market. Therefore, when the liquidity is high, the market can absorb excess demand by small price changes. On the contrary, when the liquidity is low, one unit of excess demand causes large price changes.

It should be specially mentioned that the phenomenological approach of Bouchaud and Cont (1998) succeeded in describing the dynamics of the market by identifying the demand functions of representative market participants. However, from the viewpoint of time-series analysis, making some modifications to the original model, we derive an alternative time-series model to the conventional GARCH (Generalized AutoRegressive Conditional Heteroskedastic) model proposed for financial time-series modeling by econometricians.

The purpose of this paper is twofold. First, we propose a new time-series model for describing the dynamics of financial markets. The distinctive characteristic of this model is that it considers excess demand and market liquidity as unobservable state variables. Consequently, the model is defined by a partially observed system of stochastic differential equations. Second, in order to estimate the resulting model, we utilize the local linearization method. The local linearization method is an efficient estimation method for nonlinear stochastic differential equations, developed by Ozaki (1985), and extended in subsequent studies (see Ozaki, 1992; Shoji and Ozaki, 1997, 1998; Iino and Ozaki, 2000a). In addition to its small sample performance, the local linearization method has the advantage that approximate linear difference equations can be derived, which can be estimated easily using the Kalman filter (see Ozaki, 1993).
2 The Model

We consider the following partially observed system of nonlinear stochastic differential equations:

\[
\begin{align*}
\frac{dP_t}{t} &= \lambda_t \phi_t dt + \lambda_t dW_{1,t}, \\
\frac{d\phi_t}{t} &= (a + b\phi_t) dt + cdW_{2,t}, \\
\frac{d\ln\lambda_t}{t} &= (d + e\ln\lambda_t) dt + f dW_{3,t},
\end{align*}
\]

where \(W_{1,t}, W_{2,t}, W_{3,t}\) are independent Wiener processes, \(P_t\) is asset price, \(\lambda_t\) is liquidity, \(\phi_t\) is excess demand, and \(a, b, c, d, e\) and \(f\) are constant parameters.

This model is an extension of the microstructure model, proposed by Bouchaud and Cont (1998). The main differences between the two models stem from the treatment of the excess demand, \(\phi_t\), and the liquidity, \(\lambda_t\):

1. The excess demand, \(\phi_t\), is considered as an unobserved variable instead of identifying demand functions of representative market participants.

2. The liquidity of the market, \(\lambda_t\), is also considered as an unobserved state variable that is allowed to vary over time.

3. The liquidity, \(\lambda_t\), has effects on the conditional variance as well as the conditional mean.

As a result, the above model can describe the dynamics of price more flexibly than the original model. In particular, it expresses the variation of conditional variance, the most prominent characteristic of financial markets, by the change of the liquidity of the market. Considering the market mechanism, such a relation between the liquidity and the conditional variance is more natural.
3 The Estimation Method

In order to estimate the partially observed system from discretely observed data, we develop a new estimation method by means of the local linear approximation method developed by Iino and Ozaki (2000a).

The method consists of the following steps: 1) approximating the first two conditional moments using the local linearization method; and 2) deriving a linear state-space model in discrete time, based on the assumption that the transition densities are approximated by Gaussian densities.

3.1 Local Linear Approximation of Conditional Moments

Let \( m_t^{(1)}, m_t^{(2)}, m_t^{(3)} \) denote the conditional mean of \( P_t, \phi_t, \ln \lambda_t \), respectively; then we have the following (see Appendix):

\[
\frac{dm_t^{(1)}}{dt} = E(\lambda_t \phi_t), \tag{4}
\]

\[
\frac{dm_t^{(2)}}{dt} = a + bm_t^{(2)}, \tag{5}
\]

\[
\frac{dm_t^{(3)}}{dt} = d + em_t^{(3)}. \tag{6}
\]

In a similar way, let \( V_t^{(1)}, V_t^{(2)}, V_t^{(3)} \) denote the conditional variance of \( P_t, \phi_t, \ln \lambda_t \), respectively; then we have the following (see Appendix):

\[
\frac{dV_t^{(1)}}{dt} = E(\lambda_t^2), \tag{7}
\]

\[
\frac{dV_t^{(2)}}{dt} = 2bV_t^{(2)} + c^2, \tag{8}
\]

\[
\frac{dV_t^{(3)}}{dt} = 2eV_t^{(3)} + f^2. \tag{9}
\]

Generally, it is impossible to solve nonlinear differential equations that contain expected values. However, approximating nonlinear functions locally by linear functions, we can...
derive an efficient approximation method (see Iino and Ozaki, 2000b, 2000c).

Therefore, assuming that the first and second partial derivatives of \( \lambda_s \phi_t \) and \( \lambda_t^2 \) are constant in a short interval \([s, s + \Delta s]\) for \( s < t \), we have (see Appendix)

\[
\begin{align*}
\frac{dm_t^{(1)}}{dt} & \approx -\lambda_s \phi_t \ln \lambda_s + \lambda_s m_t^{(2)} + \lambda_s \phi_t m_t^{(3)} + \frac{1}{2} \lambda_s \phi_t V_t^{(3)}, \\
\frac{dV_t^{(1)}}{dt} & \approx \lambda_s^2 (1 - 2 \ln \lambda_s) + 2 \lambda_s^2 m_t^{(3)} + 2 \lambda_s^2 V_t^{(3)}.
\end{align*}
\]  

Hence, we have the following system of differential equations:

\[
\frac{dY_t}{dt} = M_s Y_t,
\]  

where

\[
Y_t = (1 \ m_t^{(1)} \ m_t^{(2)} \ m_t^{(3)} \ V_t^{(1)} \ V_t^{(2)} \ V_t^{(3)})',
\]

\[
M_s = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
-\lambda_s \phi_t \ln \lambda_s & a & 0 & 0 & 0 & 0 \\
\lambda_s & b & 0 & 0 & 0 & 0 \\
\lambda_s^2 & e & 0 & 0 & 0 & 0 \\
\lambda_s^2 (1 - 2 \ln \lambda_s) & 2 \lambda_s^2 & 0 & 0 & 0 & 0 \\
\lambda_s^2 & 0 & 0 & 0 & 0 & 0 \\
f^2 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

Given this linear system, we obtain

\[
Y_t = \Psi_s Y_s,
\]

where

\[
\Psi_s = \exp\{M_s(t - s)\}
\]

\[
= \sum_{k=1}^{\infty} \frac{(M_s(t - s))^k}{k!},
\]

and the initial condition at \( s \) is

\[
Y_s = (1 \ P_s \ \phi_s \ \ln \lambda_s \ 0 \ 0 \ 0)',
\]
3.2 State-Space Representation

Suppose we wish to estimate the parameters, $\theta$, of the model from discrete time observations, $Z^n = \{Z_1, Z_2, \ldots, Z_n\}$.

Assuming that the transition densities are approximated by Gaussian densities, then the original nonlinear differential equations (1)-(3) are approximated by linear difference equations, the conditional mean and variance of which can be calculated by equation (13).

Noting that the conditional mean at discrete time, $t$, is the linear function of $P_{t-1}$, $\phi_{t-1}$, and $\ln \lambda_{t-1}$, we have the following state-space representation:

$$X_t = A_{t-1}X_{t-1} + B_{t-1} + \varepsilon_t,$$  \hspace{1cm} (14)

$$Z_t = H_t X_t,$$  \hspace{1cm} (15)

where

$$X_t = (P_t \phi_t \ln \lambda_t)',$$

$$A_t = [\Psi_t^{(ij)}], \quad 2 \leq i, j \leq 4,$$

$$B_t = [\Psi_t^{(i)}], \quad i = 1, 2 \leq j \leq 4,$$

$$H_t = (1 \hspace{0.5cm} 0 \hspace{0.5cm} 0),$$

$$Q_t = \begin{pmatrix} V_t^{(1)} & 0 & 0 \\ 0 & V_t^{(2)} & 0 \\ 0 & 0 & V_t^{(3)} \end{pmatrix},$$

and $\Psi_t^{(ij)}$ is the $(ij)$-th component of $\Psi_t$ calculated in the previous subsection.

Given this linear-state space representation, the parameters, $\theta$, are estimated by maximizing the log likelihood function, calculated as follows (see Balakrishnan, 1987):

$$L(\theta) = -\frac{1}{2} \sum_{t=1}^{n} \left\{ \frac{(Z_t - H_t(A_tX_{t-1} + B_{t-1}))^2}{P_t^{-}}} + \ln(2\pi P_t^{-}) \right\},$$

6
and $\hat{X}_t$, $P_t^-$ are calculated recursively by using the Kalman filtering procedure:

$$\hat{X}_t = (A_{t-1}\hat{X}_{t-1} + B_{t-1})$$
$$+ K_t (Z_t - H_t(A_{t-1}\hat{X}_{t-1} + B_{t-1})),
K_t = P_t^{-}H_t'(H_tP_t^{-}H_t')^{-1},
$$
$$P_t^- = A_{t-1}P_{t-1}A_{t-1}' + Q_{t-1},
$$
$$P_t = P_t^- - K_tH_tP_t^-.$$

### 3.3 Numerical Studies

In this subsection, the proposed model is fitted to financial time-series data. We analyze the daily price of the Japanese Yen/US Dollar exchange rate from 1994:1 to 1997:12. The observation data, $Z_t$, is computed as follows:

$$Z_t = \ln(FX_t) \times 100,$$

where $FX_t$ denotes the exchange rate at time, $t$. The transition of the observation data is shown in Figure 1. The resulting estimates of the excess demand, $\phi_t$, and the liquidity, $\lambda_t$, are given in Figure 2 and Figure 3, respectively.

In order to evaluate the performance of the proposed model, we also estimate the GARCH(1,1) model frequently used in the financial literature:

$$Z_t = a + bZ_{t-1} + \xi_t, \quad \xi_t \sim N(0, \sigma_t^2),$$
$$\sigma_t^2 = c + d\sigma_{t-1}^2 + e\xi_{t-1}^2.$$

Table 1 reports the estimates of the parameters. Table 2 shows the results from the diagnostic checks on the two models. Specifically, the mean square error (MSE), log likelihood (Likelihood), Akaike Information Criterion (AIC), skewness and kurtosis of normalized prediction errors are computed. The distributions of the normalized prediction errors are
shown in Figure 4 and Figure 5. Examining Table 2, we see the superiority of the proposed model over the GARCH model, because the former consistently shows good performance. In particular, noting that the AIC is 1,949 for the proposed model and 1,996 for the GARCH model, the difference between the two models is evident.

4 Conclusion

In this paper, a nonlinear time-series model, based on microstructure theory, is proposed. The distinctive characteristic of the model is that it considers excess demand and liquidity as unobserved state variables. A method of estimation for the resulting partially observed system is discussed. The performance of the proposed model is evaluated by empirical data analysis. As a result, its superiority over the conventional GARCH model is demonstrated. Consequently, we conclude that the proposed model has desirable properties from both theoretical and empirical viewpoints.
Appendix

To simplify the exposition, in this appendix we consider the following n-dimensional stochastic differential equation:

\[ dX_t = \mu(X_t)dt + \sigma(X_t)dW_t \]

equivalently in coordinate form

\[ dX^{(i)}_t = \mu^{(i)}(X_t)dt + \sigma^{(i)}(X_t)dW^{(i)}_t, \quad i = 1, \ldots, n \] (16)

where \( X_t = (X^{(1)}_t, X^{(2)}_t, \ldots, X^{(n)}_t)' \) is an n-dimensional stochastic process, \( W_t = (W^{(1)}_t, W^{(2)}_t, \ldots, W^{(n)}_t) \) is an n-dimensional standard Brownian motion, and \( \mu(X_t), \sigma(X_t) \) are n-dimensional vector valued functions.

Our first task is to derive the differential equations describing the dynamics of the conditional moments. Let \( m^{(i)}_t, V^{(i)}_t \) denote the conditional mean and the conditional variance of \( X^{(i)}_t \), respectively,

\[ m^{(i)}_t = E[X^{(i)}_t], \quad V^{(i)}_t = E[(X^{(i)}_t - m^{(i)}_t)^2]. \]

Taking the expected value in the integral form of (16),

\[ \frac{dm^{(i)}_t}{dt} = E[\mu^{(i)}(X_t)]. \] (17)

Hence, we have

\[ \frac{dm^{(i)2}_t}{dt} = 2m^{(i)}_t E[\mu^{(i)}(X_t)]. \]

By Ito’s formula (see, e.g., Karatzas and Shreve, 1991),

\[ dX^{(i)2}_t = \left(2\mu^{(i)}(X_t)X^{(i)}_t + \sigma^{(i)}(X_t)^2\right)dt + 2X^{(i)}_t\sigma^{(i)}(X_t)dW^{(i)}_t. \]

9
Let $M_t^{(i)} = E[X_t^{(i)}^2]$. Then, taking the expected value,

$$
\frac{dM_t^{(i)}}{dt} = 2E[\mu^{(i)}(X_t)X_t^{(i)}] + E[\sigma^{(i)}(X_t)^2].
$$

Noting $V_t^{(i)} = M_t^{(i)} - m_t^{(i)}$, we have

$$
\frac{dV_t^{(i)}}{dt} = 2Cov(\mu^{(i)}(X_t), X_t^{(i)}) + E[\sigma^{(i)}(X_t)^2].
$$

(18)

We next approximate the resulting nonlinear differential equations (17), (18) by linear ones. By Ito’s formula,

$$
\mu^{(i)}(X_t) = \mu^{(i)}(X_s) + \sum_{k=1}^n \int_s^t \frac{\partial \mu^{(i)}}{\partial x^{(k)}} (X_u) dX_u^{(k)} + \frac{1}{2} \sum_{k=1}^n \int_s^t \frac{\partial^2 \mu^{(i)}}{\partial x^{(k)2}} (X_u) \sigma^{(k)}(X_u)^2 du.
$$

Assuming $\frac{\partial \mu^{(i)}}{\partial x^{(k)}} (X_s)$ and $\frac{\partial^2 \mu^{(i)}}{\partial x^{(k)2}} (X_s)$ are constant in a short interval $[s, s + \Delta s]$ for $s < t$,

$$
\mu^{(i)}(X_t) \approx \mu^{(i)}(X_s) + \sum_{k=1}^n \frac{\partial \mu^{(i)}}{\partial x^{(k)}} (X_s) (X_t^{(k)} - X_s^{(k)}) + \frac{1}{2} \sum_{k=1}^n \frac{\partial^2 \mu^{(i)}}{\partial x^{(k)2}} (X_s) \int_s^t \sigma^{(k)}(X_u)^2 du.
$$

Taking the expectation, and noting $V_t^{(i)} = E[\int_s^t \sigma^{(i)}(X_u)^2 du]$,

$$
E[\mu^{(i)}(X_t)] \approx \mu^{(i)}(X_s) + \sum_{k=1}^n \frac{\partial \mu^{(i)}}{\partial x^{(k)}} (X_s) (m_t^{(k)} - m_s^{(k)}) + \frac{1}{2} \sum_{k=1}^n \frac{\partial^2 \mu^{(i)}}{\partial x^{(k)2}} (X_s) V_t^{(k)}.
$$

In a similar way,

$$
E[\sigma^{(i)}(X_t)^2] \approx \sigma^{(i)}(X_s)^2 + \sum_{k=1}^n \frac{\partial \sigma^{(i)}}{\partial x^{(k)}} (X_s) (m_t^{(k)} - m_s^{(k)}) + \frac{1}{2} \sum_{k=1}^n \frac{\partial^2 \sigma^{(i)}}{\partial x^{(k)2}} (X_s) V_t^{(k)}.
$$

Furthermore,

$$
Cov(\mu^{(i)}(X_t), X_t^{(i)}) = E[(\mu^{(i)}(X_t) - E(\mu^{(i)}(X_t)))(X_t^{(i)} - E(X_t^{(i)}))] = E\left[ \left( \sum_{k=1}^n \int_s^t \frac{\partial \mu^{(i)}}{\partial x^{(k)}} (X_u) \sigma^{(k)}(X_u) dW_u^{(k)} \right) \left( \int_s^t \sigma^{(i)}(X_u) dW_u^{(i)} \right) \right] = E\left[ \int_s^t \frac{\partial \mu^{(i)}}{\partial x^{(i)}} (X_u) \sigma^{(i)}(X_u)^2 du \right] \approx \frac{\partial \mu^{(i)}}{\partial x^{(i)}} (X_s) E\left[ \int_s^t \sigma^{(i)}(X_u)^2 du \right] = \frac{\partial \mu^{(i)}}{\partial x^{(i)}} (X_s) V_t^{(i)}.
$$
Hence, we have the following approximate linear differential equations:

\[
\frac{dm^{(i)}}{dt} = \left\{ \mu^{(i)}(X_s) - \sum_{k=1}^{n} \frac{\partial \mu^{(i)}}{\partial x^{(k)}}(X_s)m^{(k)}_s \right\} \\
+ \sum_{k=1}^{n} \frac{\partial \mu^{(i)}}{\partial x^{(k)}}(X_s)m^{(k)}_t + \frac{1}{2} \sum_{k=1}^{n} \frac{\partial^2 \mu^{(i)}}{\partial x^{(k)}^2}(X_s)V^{(k)}_t, \quad (19)
\]

\[
\frac{dV^{(i)}}{dt} = \left\{ \sigma^{(i)}(X_s)^2 - \sum_{k=1}^{n} \frac{\partial \sigma^{(i)}}{\partial x^{(k)}}(X_s)m^{(k)}_s \right\} \\
+ \sum_{k=1}^{n} \frac{\partial \sigma^{(i)}}{\partial x^{(k)}}(X_s)m^{(k)}_t + \frac{1}{2} \sum_{k=1}^{n} \frac{\partial^2 \sigma^{(i)}}{\partial x^{(k)}^2}(X_s)V^{(k)}_t + 2\frac{\partial \mu^{(i)}}{\partial x^{(i)}}(X_s)V^{(i)}_t. \quad (20)
\]
References


Captions of Figures

Figure 1. Transition of the observation data $Z_t$.

Figure 2. Estimates of the excess demand $\phi_t$.

Figure 3. Estimates of the liquidity $\lambda_t$.

Figure 4. Distribution of normalized prediction errors for the proposed model.

Figure 5. Distribution of normalized prediction errors for the GARCH model.
### Table 1. Parameter estimates

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### Table 2. Diagnosis

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