Linear estimation of continuous-discrete linear state space models with multiplicative noise

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J.C. Jimenez* and T. Ozaki †

Abstract

This paper deals with the estimation of the state variable of continuous-discrete linear state space models with multiplicative noise. Specifically, the optimal minimum variance linear filter for that class of models is constructed. Moreover, the solutions of the differential equations that describe the evolution of the two first conditional moments between observations are obtained and an algorithm for their numerical computation is also given. The performance of the linear filter is illustrated by mean of numerical experiments.

1 Introduction

The estimation of the state of a continuous stochastic dynamical system from noise discrete observations taken on the state is of central importance to solve diverse scientific and technological problems. The major contribution to the solution of this estimation problem is due to Kalman [10]-[12], who provided a sequential and computationally efficient solution to the optimal filtering and prediction problem for linear state space models with additive noise.

Unfortunately, this Kalman algorithm is restricted to the class of continuous linear systems with additive noise. So that, it is not applicable to the large variety of estimation problems that involve linear stochastic differential equations (SDEs) with multiplicative noise. It includes important problems in chemistry, biology, ecology, economics, physics and engineering (see [13, 15] and references in therein).

It is well known that, for linear models with multiplicative noise, the state is a non Gaussian process [18] and that the optimal filter is nonlinear and very difficult to compute in practice [14]. This is the reason for which, for this kind of models, the construction of

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high quality suboptimal filters that approximate the optimal one becomes very important.
Prominent examples of suboptimal filters are the linear ones, which have been widely used
for the estimation of the state of both, continuous-continuous [14, 16, 18] and discrete-
have also been proposed for the discrete-discrete models. However, no suboptimal filters
for continuous-discrete models with multiplicative noise have been considered.

The proposal of this paper is to construct an optimal linear filter for this class of
models.

The paper is organized as follows. In section 2, the optimal minimum variance linear
filter is introduced. In section 3, the solutions of the differential equations that describe
the evolution of the two first conditional moments between observations are obtained
and, in section 4, an algorithm for their numerical computation is presented. In the last
section, the effectiveness of the filter is illustrated by mean of numerical experiments.

2 Optimal minimum variance linear filter

Let the linear state space model defined by the continuous state equation

\[
dx(t) = (A(t)x(t) + a(t))dt + \sum_{i=1}^{m}(B_i(t)x(t) + b_i(t))dw^i(t),
\]
and the discrete observation equation

\[
z_{tk} = C(t_k)x(t_k) + \sum_{i=1}^{n}D_i(t_k)x(t_k)e_i^t + F(t_k)e_{tk}, \text{ for } k = 0, 1, \ldots, N
\]

where \(x(t) \in \mathbb{R}^d\) is the state vector at the instant of time \(t\), \(z_{tk} \in \mathbb{R}^r\) is the measurement
vector at the instant of time \(t_k\), \(w\) is an \(m\)-dimensional Wiener process with independent
components, and \(\{e_i^t : e_{tk} \sim N(0, \Lambda), \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)\}\) and \(\{e_{tk} : e_{tk} \sim N(0, \Sigma)\}\)
are sequences of random vectors i.i.d.

Let \(\hat{x}_{t/t} = \mathcal{E}(x(t)/Z_t)\) and \(P_{t/t} = \mathcal{E}((x(t) - \hat{x}_{t/t})^2/Z_t)\), where \(\mathcal{E}(.)\) denotes expected
value and \(Z_t = \{z_{tk} : t_k \leq t\}\).

Suppose that \(E(w(t)w^t(t)) = I\), \(E(e_i^t e_{tk}) = \theta_i^t(t_k)\), \(\hat{x}_{0/0} = \mathcal{E}(x(0)/Z_0) < \infty\) and \(P_0/0 = \mathcal{E}((x(0) - \hat{x}_{0/0})^2/Z_0) < \infty\).

**Theorem 1** The optimal (minimum variance) linear filter for the linear model (1)-(2)
consists of equations of evolution for the conditional mean \(\hat{x}_{t/t}\) and covariance matrix \(P_{t/t}\).
Between observations, these satisfy the ordinary differential equations

\[
dx_{t/t} = (A(t)\hat{x}_{t/t} + a(t))dt
\]
and

\[
P_{t/t} = (A(t)P_{t/t} + P_{t/t}A(t) + \sum_{i=1}^{m}b_i(t)b_i^t(t)
\]
\[
\begin{align*}
&+ \sum_{i=1}^{m} B_i(t) \left( P_{t/t} + \tilde{x}_{t/t} \tilde{x}_{t/t}^T \right) B_i(t) \\
&+ \sum_{i=1}^{m} \{B_i(t) b_i(t) \tilde{x}_{t/t}^T + b_i(t) \tilde{x}_{t/t}^T B_i^T(t) \} dt
\end{align*}
\]

for all \( t \in [t_k, t_{k+1}) \). At an observation at \( t_k \), they satisfy the difference equation

\[
\hat{x}_{t_{k+1}/t_k} = \hat{x}_{t_{k+1}/t_k} + K_{t_{k+1}} (z_{t_{k+1}} - C(t_{k+1}) \hat{x}_{t_{k+1}/t_k}),
\]

\[
P_{t_{k+1}/t_k} = P_{t_{k+1}/t_k} - K_{t_{k+1}} C(t_{k+1}) P_{t_{k+1}/t_k}.
\]

where

\[
K_{t_{k+1}} = P_{t_{k+1}/t_k} C^T(t_{k+1}) (C(t_{k+1}) P_{t_{k+1}/t_k} C^T(t_{k+1}) + F(t_{k+1}) \Sigma F^T(t_{k+1})
\]

\[
+ \sum_{i=1}^{n} \lambda_i D_i(t_{k+1}) (P_{t_{k+1}/t_k} + \tilde{x}_{t_{k+1}/t_k} \tilde{x}_{t_{k+1}/t_k}^T) D_i^T(t_{k+1})
\]

\[
+ \sum_{i=1}^{n} D_i(t_{k+1}) \tilde{x}_{t_{k+1}/t_k} (\Theta^i(t_{k+1}))^T F(t_{k+1})
\]

\[
+ \sum_{i=1}^{n} F(t_{k+1}) \Theta^i(t_{k+1}) \tilde{x}_{t_{k+1}/t_k} D_i^T(t_{k+1})^{-1}
\]

is the filter gain. The predictions \( \hat{x}_{t/p} \) and \( P_{t/p} \) are accomplished, respectively, via expressions (3) and (4) with initial conditions \( \hat{x}_{p/p} \) and \( P_{p/p} \) and \( p < t \).

\textbf{Proof.} Obviously, in absence of observations (i.e., between observations) it is holds that \( \mathcal{E}(x(t)) = \mathcal{E}(x(t)/Z_t) = \mathcal{E}(x(t)/Z_r) \), for all \( t \in [t_k, t_{k+1}) \) and \( r < t \). Moreover, it is well known that in this case the conditional mean is the minimum variance estimate for the filtering and prediction problem [7].

The equation (3) for the conditional means follows taking expectation on both sides of the integral form of (1).

By definition the differential of the conditional covariance matrix is

\[
dP_{t/t} = d(\mathcal{E}(x_{t/t} x_{t/t}^T)) - d(\hat{x}_{t/t} \hat{x}_{t/t}^T).
\]

Using the Ito formula it is obtained that

\[
d(\mathcal{E}(x_{t/t} x_{t/t}^T)) = \{A(t) \mathcal{E}(x_{t/t} x_{t/t}^T) + \mathcal{E}(x_{t/t} x_{t/t}^T) A(t) + a(t) \hat{x}_{t/t} + \hat{x}_{t/t} a^T(t)
\]

\[
+ \sum_{i=1}^{m} B_i(t) \mathcal{E}(x_{t/t} x_{t/t}^T) B_i(t)
\]

\[
+ \sum_{i=1}^{m} (B_i(t) b_i(t) \hat{x}_{t/t} + b_i(t) \hat{x}_{t/t} B_i^T(t) + b_i(t) b_i^T(t)) \} dt,
\]

\[
dP_{t/t} = a(t) \hat{x}_{t/t} + \hat{x}_{t/t} a^T(t) dt
\]
and from expression (3) is obtained that

\[ d(\tilde{x}_{t_0/t_0}^t + \tilde{x}_{t_0/t_0}^t A(t) + a(t)\tilde{x}_{t_0/t_0}^t + \tilde{x}_{t_0/t} a^T(t)\) dt. \] (10)

Thus, the expression (4) follows from substituting (9) and (10) in (8).

At an observation at \( t_{k+1} \), let consider the following unbiased filter estimate

\[ \hat{x}_{t_{k+1}/t_k} = \hat{x}_{t_{k+1}/t_k} + K_{t_{k+1}}(x_{t_{k+1}} - C(t_{k+1})\hat{x}_{t_{k+1}/t_k}), \] (11)

where \( K_{t_{k+1}} \) is the matrix that minimizes the functional

\[ \mathcal{E}(\hat{x}_{t_{k+1}}^T U \hat{x}_{t_{k+1}}), \] (12)

where \( \hat{x}_{t_{k+1}} = x_{t_{k+1}} - \hat{x}_{t_{k+1}/t_k} \) and \( U \) is a symmetric matrix.

By substituting (2) in (11) it is obtained that

\[ \mathcal{E}(\hat{x}_{t_{k+1}}^T \hat{x}_{t_{k+1}}) = (I - K_{t_{k+1}} + C(t_{k+1}))^T \mathcal{E}(P_{t_{k+1}/t_k}) (I - K_{t_{k+1}} + C(t_{k+1}))^T \]
\[ + \sum_{i=1}^{n} K_{t_{k+1}} D_i(t_{k+1}) \mathcal{E}(P_{t_{k+1}/t_k} + \hat{x}_{t_{k+1}/t_k} \hat{x}_{t_{k+1}/t_k}^T) D_i^T(t_{k+1}) K_{t_{k+1}} \]
\[ + \sum_{i=1}^{n} D_i(t_{k+1}) \mathcal{E}(\hat{x}_{t_{k+1}/t_k})(\theta^i(t_{k+1}))^T F^T(t_{k+1}) K_{t_{k+1}} \]
\[ + \sum_{i=1}^{n} K_{t_{k+1}} F(t_{k+1}) \theta^i(t_{k+1}) \mathcal{E}(\hat{x}_{t_{k+1}/t_k}^T) D_i^T(t_{k+1}) K_{t_{k+1}} \]
\[ + K_{t_{k+1}} F(t_{k+1}) \Sigma F^T(t_{k+1}) K_{t_{k+1}} \] (13)

Using that \( \mathcal{E}(\hat{x}_{t_{k+1}}^T U \hat{x}_{t_{k+1}}) = tr(\mathcal{E}(\hat{x}_{t_{k+1}}^T) U) \) it follows that

\[ \partial \mathcal{E}(\hat{x}_{t_{k+1}}^T U \hat{x}_{t_{k+1}})/\partial K_{t_{k+1}} = 2UK_{t_{k+1}}(C(t_{k+1})^\top \mathcal{E}(P_{t_{k+1}/t_k})^\top C(t_{k+1}) + F(t_{k+1}) \Sigma F^T(t_{k+1})) \]
\[ + \sum_{i=1}^{n} K_{t_{k+1}} D_i(t_{k+1}) \mathcal{E}(P_{t_{k+1}/t_k} + \hat{x}_{t_{k+1}/t_k} \hat{x}_{t_{k+1}/t_k}^T) D_i^T(t_{k+1}) \]
\[ + \sum_{i=1}^{n} D_i(t_{k+1}) \mathcal{E}(\hat{x}_{t_{k+1}/t_k})(\theta^i(t_{k+1}))^T F^T(t_{k+1}) \]
\[ + \sum_{i=1}^{n} F(t_{k+1}) \theta^i(t_{k+1}) \mathcal{E}(\hat{x}_{t_{k+1}/t_k}^T) D_i^T(t_{k+1}) \]
\[ - 2U \mathcal{E}(P_{t_{k+1}/t_k}) C^\top(t_{k+1}). \]

Taking into account some of trivial properties of the expectation operator \( \mathcal{E} \) and that \( \mathcal{E}(P_{t_{k+1}/t_k}) \neq 0 \) then, the expression (7) for \( K_{t_{k+1}} \) is finally obtained by solving

\[ \partial \mathcal{E}(\hat{x}_{t_{k+1}}^T U \hat{x}_{t_{k+1}})/\partial K_{t_{k+1}} = 0. \]
The expression (6) is obtained by substituting (7) in (13) and using that \( \mathcal{E}(\hat{x}_{t+1|k}, \hat{x}_{t+1}^\top) = \mathcal{E}(P_{t+1|k+1}) \). □

Note that, for linear models (1)-(2) with \( a(t) \equiv 0 \) and \( B_i(t) \equiv D_j(t) \equiv 0 \) (for all \( i = 1, \ldots, m \) and \( j = 1, \ldots, n \)), the filter (3)-(6) reduces to the (optimal minimum variance) Kalman-Busy filter for continuous-discrete models with additive noise [10]-[12].

3 Solution of the equations for the first two conditional moments between observations

This section deals with the problem of solving the ordinary differential equations that appear in Theorem 1. Specifically, the solution of the equations for the predictions \( \hat{x}_{t/k} \) and \( P_{t/k} \) will be given.

Here, the symbols \( \text{vec}, \oplus \) and \( \otimes \) will denote the vectorization operator, the Kronecker sum and product, respectively.

**Theorem 2** The general solution of the system of differential equations (3)-(4) is given by

\[
\hat{x}_{t/k} = \exp\left( \int_0^{t-t_k} A(u + t_k)du \right) x_{t_k/k} + \int_0^{t-t_k} \exp\left( - \int_0^s A(u + t_k)du \right) a(s + t_k) \, ds \tag{14}
\]

and

\[
\text{vec}(P_{t/k}) = \exp\left( \int_0^{t-t_k} A(u + t_k)du \right) \text{vec}(P_{t_k/k}) + \int_0^{t-t_k} \exp\left( - \int_0^s A(u + t_k)du \right) B(s + t_k)ds,
\]

for all \( t \in [t_k, t_{k+1}] \), where

\[
A(s) = A(s) \oplus A(s) + \sum_{i=1}^m B_i(s) \otimes B_i(s)
\]

and

\[
B(s) = \sum_{i=1}^m (B_i(s) \otimes B_i(s)) \text{vec}(\hat{x}_{s/k} \hat{x}_{s/k}^\top) + \sum_{i=1}^m \text{vec}(b_i(s) \otimes b_i(s))
\]

\[
+ \sum_{i=1}^m (b_i(s) \otimes B_i(s) + B_i(s) \otimes b_i(s)) \text{vec}(\hat{x}_{s/k}).
\]

In particular, if \( A(t) = A, B(t) = B, a(t) = a \) and \( b(t) = b \) for \( t \), then the above solution reduces to

\[
\hat{x}_{t/k} = x_{t_k/k} + \int_0^{t-t_k} \exp(-As)ds \left( A\hat{x}_{t_k/k} + a \right) \tag{16}
\]
and
\[
\vec(P_{t/t_k}) = \exp(A(t - t_k))\vec(P_{t_k/t_k}) + \sum_{i=1}^{5} \int_0^{t-t_k} \exp(-As)B_i \exp(A_s)C_ids,
\]
(17)

where \(A = A \oplus A + \sum_{i=1}^{m} B_i \otimes B_i\), and \(A_i, B_i\) and \(C_i\) are the matrices defined in Table I.

The following lemma will be useful to demonstrate the above theorem.

**Lemma 3** (Van Loan 1978, [20]) Let \(A_1, A_2\) and \(A_3\) be square matrices, \(n_1, n_2\) and \(n_3\) be positive integers, and set \(m\) to be their sum. If the \(m \times m\) block triangular matrix \(C\) is defined by
\[
C = \begin{bmatrix} A_1 & B_1 & C_1 \\ 0 & A_2 & B_2 \\ 0 & 0 & A_3 \end{bmatrix}
\]
(18)

then, for \(s \geq 0\),
\[
\begin{bmatrix} F_1(s) & G_1(s) & H_1(s) \\ 0 & F_2(s) & G_2(s) \\ 0 & 0 & F_3(s) \end{bmatrix} = \exp(sC)
\]
(19)

where
\[
F_j(s) = \exp(A_j s), \text{ for } j = 1, 2, 3
\]
\[
G_j(s) = \int_0^s \exp(A_j(s - u))B_j \exp(A_{j+1}u)du, \text{ for } j = 1, 2
\]
\[
H_1(s) = \int_0^s \exp(A_1(s - u))C_1 \exp(A_3u)du
\]
\[
+ \int_0^s \int_0^u \exp(A_1(s - u))B_1 \exp(A_2(u - r))B_2 \exp(A_3r)drdu.
\]

**Proof of Theorem 2.** The solution (14) of the linear equation (3) is very well known [1]. Likewise, (15) is also solution of a linear equation. Specifically, it is solution of
\[
dvec(P_{t/t_k}) = (B(t) + A(t)\vec(P_{t/t_k}))dt,
\]
which is obtained by applying the operator \(\vec\) to the equation (4).

In the case that \(A(t) = A, B(t) = B, a(t) = a\) and \(b(t) = b\) for \(t\), the expressions (14) and (15) reduce to
\[
\hat{x}_{t/t_k} = \exp(A(t - t_k))x_{t_k/t_k} + \int_0^{t-t_k} \exp(-As) a ds
\]
(20)
and

\[ \text{vec}(\mathbf{P}_{t/t_k}) = \exp(A(t-t_k)) \text{vec}(\mathbf{P}_{t_k/t_k}) + \int_0^{t-t_k} \exp(-\dot{A}s) B(s + t_k) ds, \]  

(21)

respectively.

The expression (16) is obtained from (20) and the identity \( \int_0^{t-t_k} \exp(-\dot{A}s) ds = -(\exp(-A(t-t_k)) - I). \)

Using lemma 3, the expression (16) can be rewritten as

\[ \hat{x}_{t/t_k} = \hat{x}_{t_k/t_k} + L^T \exp((t-t_k)C) R, \]  

(22)

where

\[ C = \begin{bmatrix} A A \hat{x}_{t_k/t_k} + a \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{(d+1) \times (d+1)}, \]

and \( L^T = [I_d \ 0_d] \) and \( R^T = [0_{1 \times d} \ 1] \). Then, substituting (22) in (21), and after some algebraic manipulations the expression (17) is obtained. \( \blacksquare \)

4 Numerical computation of the prediction estimates

In the previous section, expressions for the prediction \( \hat{x}_{t/t_k} \) and \( \mathbf{P}_{t/t_k} \) were obtained. However, in general, these expressions are difficult to implement in practice by means of a computer program. In this section, simple algebraic expressions are introduced as an computationally feasible alternative to (16) and (17).

**Theorem 4** The expressions (16) and (17) are equivalent to

\[ \hat{x}_{t/t_k} = \hat{x}_{t_k/t_k} + \mathbf{g}(t), \]  

(23)

and

\[ \text{vec}(\mathbf{P}_{t/t_k}) = \mathcal{F}(t) \text{vec}(\mathbf{P}_{t_k/t_k}) + \sum_{i=1}^{5} \mathcal{G}_i(t) C_i, \]  

(24)

respectively. Here, the vector \( \mathbf{g}(t) \) and the matrices \( \mathcal{F}_i(t) \) and \( \mathcal{G}_i(t) \) are defined by matrix identities

\[
\begin{bmatrix}
F(t) & g(t) \\
0 & 1
\end{bmatrix} = \exp((t-t_k)C)
\]

\[
\begin{bmatrix}
\mathcal{F}_i(t) & \mathcal{G}_i(t) \\
0 & \mathcal{F}_i(t)
\end{bmatrix} = \exp((t-t_k)\mathcal{J}_i)
\]

where

\[ C = \begin{bmatrix} A A \hat{x}_{t_k/t_k} + a \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{(d+1) \times (d+1)}, \]
\[ \mathcal{J}_i = \begin{bmatrix} A & B_i \\ 0 & A_i \end{bmatrix}, \]

and the matrices \( A, A_i, B_i, \) and \( C_i \) are defined as in Theorem 2.

**Proof.** The expression (23) is straightly derived rewriting (16) as

\[ \hat{x}_{t/t_k} = x_{t_k/t_k} + \int_{0}^{t-t_k} \exp(A(t-t_k-u))du \left( A\hat{x}_{t_k/t_k} + a \right), \]

and using the Lemma 3, with \( s = t-t_k, A_1 = A, B_1 = A\hat{x}_{t_k/t_k} + a, \) and \( A_2 = 0 \) in (18).

The expression (24) is obtained by using Lemma 3, with \( s = t-t_k, A_1 = A, B_1 = B_i, \)
and \( A_2 = A_i \) in (18). Obviously, \( \mathcal{F}_1(t) = \exp((t-t_k)A). \]

In this way, the numerical computation of the conditional moments \( \hat{x}_{t/t_k} \) and \( P_{t/t_k} \)
is reduced to use an convenient algorithm to compute matrix exponentials, e.g., those based on rational Padé approximations [5], the Schur decomposition [5] or Krylov subspace methods [6] (for a recent review see [19]). The selection of one of them will mainly depend of the size and structure of the matrix \( A \). In many case, it is enough to use the algorithm developed in [20] which takes advantage of the special structure of the matrices \( C \) and \( \mathcal{J}_i \). However, for state space models than involve more that four state equation the differential equation (15) becomes large. In this case the Krylov subspace methods provide the more efficient and accurate algorithms to compute matrix exponentials (see [8] for more details).

5 Numerical experiments

In this section, the performance of the linear filter introduced in this paper is illustrated by means of some numerical experiments that involve an autonomous linear state space model. The values of the first two conditional moments \( \hat{x}_{t_k/t_k} \) and \( P_{t_k/t_k} \) at the instant of time \( t_k \) were computed by the equations (5)-(6) using the values of \( \hat{x}_{t_{k+1}/t_k} \) and \( P_{t_{k+1}/t_k} \) computed by the expression (23) and (24). The simulation programs were developed in Matlab version 5.3.1.

Consider us the linear model with state equation

\[ dx = \begin{pmatrix} -5 & 1 \\ 2 & -6 \end{pmatrix} x + \begin{pmatrix} 1 \\ 2 \end{pmatrix} dt + \begin{pmatrix} -0.25 & 0 \\ 0.25 & 0 \end{pmatrix} x + \begin{pmatrix} 0.5 \\ 0.25 \end{pmatrix} dw_1 + \begin{pmatrix} 0 & 0.5 \\ 0 & -0.75 \end{pmatrix} x + \begin{pmatrix} 0.5 \\ 0.25 \end{pmatrix} dw_2 \]

and observation equation

\[ z_{t_k} = x_1(t_k) + x_1(t_k) \xi_{t_k} + e_{t_k}, \]

where \( \{ \xi_{t_k} : \xi_{t_k} \sim \mathcal{N}(0, \lambda^2) \} \) and \( \{ e_{t_k} : e_{t_k} \sim \mathcal{N}(0, \sigma^2) \} \) are sequences of random vectors i.i.d; and \( \xi(\xi_{t_k}, e_{t_k}) = 0. \)
Figure 1 shows a realization of the solution of the state equation (25) computed by the explicit Euler-Maruyama scheme [13] at the instant of time $\tau_j = j\Delta$, with $\Delta = 0.0001$ and $j = 0, \ldots, 4000$.

Figure 2 shows a realization of the observed equation (26) with $\lambda^2 = 0$ and $\sigma^2 = 100$. Figures 3 and 4 present, respectively, the first and second exact conditional moments together the estimated $\hat{x}_{t_h/t_h}$ and $P_{t_h/t_h}$. $\hat{x}_{t_h/t_h}$ and $P_{t_h/t_h}$ were computed from the set of observations $\{z_{t_k}\}_{k=1}^{N}$, with $t_k = kh$, $h = 0.01$ and $N = 400$.

In Figure 5, a realization of the observed equation (26) with $\lambda^2 = 600$ and $\sigma^2 = 0$ is displayed. Figures 6 and 7 present, respectively, the first and second exact conditional moments together the estimated $\hat{x}_{t_h/t_h}$ and $P_{t_h/t_h}$.

Figure 8 shows a realization of the observed equation (26) with $\lambda^2 = 600$ and $\sigma^2 = 100$. Figures 9 and 10 display, respectively, the first and second exact conditional moments together estimated $\hat{x}_{t_h/t_h}$ and $P_{t_h/t_h}$.

Note that, in all the cases, there is no significative difference between the exact and the estimated values of the two first conditional moments, which illustrate the effectiveness of the linear filter.

6 Conclusion

The optimal minimum variance linear filter for linear continuous-discrete models with multiplicative noise was constructed and their effectiveness was illustrated by mean of numerical experiments. Moreover, this linear filter constitutes the kernel of the Local Linearization filters for nonlinear continuous-discrete models with multiplicative noise [9], which have been successful applied to solve several financial estimation problems.

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References

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<td>2</td>
<td>$I_d \otimes C$</td>
<td>$\sum_{i=1}^{m} (B_i \otimes B_i)(I_d \otimes L_i^\top)$</td>
<td>$\text{vec}(R \hat{x}_{ts/ta}^\top)$</td>
</tr>
<tr>
<td>3</td>
<td>$C \otimes I_d$</td>
<td>$\sum_{i=1}^{m} (B_i \otimes B_i)(L_i^\top \otimes I_d)$</td>
<td>$\text{vec}(\hat{x}_{ts/ta}^\top R_i^\top)$</td>
</tr>
<tr>
<td>4</td>
<td>$C \otimes I_{d+1}$</td>
<td>$\sum_{i=1}^{m} (b_i \otimes B_i + B_i \otimes b_i)(L_i^\top \otimes R_i^\top)$</td>
<td>$\text{vec}(I_{d+1})$</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>$\sum_{i=1}^{m} (B_i \otimes B_i)\text{vec}(\hat{x}<em>{ts/ta}^\top \hat{x}</em>{ts/ta}^\top) + (b_i \otimes B_i + B_i \otimes b_i)\text{vec}(\hat{x}_{ts/ta}^\top) + \text{vec}(b_i \otimes b_i^\top)$</td>
<td>1</td>
</tr>
</tbody>
</table>

Table I: Values of the constants $\mathcal{A}_i$, $B_i$ and $C_i$ of the expression (17). Here, $C = \begin{bmatrix} A \ A \hat{x}_{ts/ta}^\top + a \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^{(d+1) \times (d+1)}$, $L_i^\top = [I_d \ 0_d] \in \mathbb{R}^{d \times (d+1)}$, $R_i^\top = [0_{1 \times d} \ 1] \in \mathbb{R}^{1 \times (d+1)}$, and $I_m$ is the $m \times m$ identity matrix.
Legends of Figures:

Figure 1: realization of the solution of the state equation (25) computed by the explicit Euler-Maruyama scheme. Top: $x_1$; Bottom: $x_2$.

Figure 2: realization of the observed equation (26) with $\lambda^2 = 0$ and $\sigma^2 = 100$.

Figure 3: first exact conditional moment together the estimated $\hat{x}_{t_k/t_k}$, given the observation of the Fig. 2.

Figure 4: second exact conditional moments together the estimated $P_{t_k/t_k}$, given the observation of the Fig. 2. Top: $[P_{t_k/t_k}]_{11}$; Middle: $[P_{t_k/t_k}]_{22}$; Bottom: $[P_{t_k/t_k}]_{12}$.

Figure 5: realization of the observed equation (26) with $\lambda^2 = 600$ and $\sigma^2 = 0$.

Figure 6: first exact conditional moment together estimated $\hat{x}_{t_k/t_k}$, given the observation of the Fig. 5.

Figure 7: second exact conditional moments together estimated $P_{t_k/t_k}$, given the observation of the Fig. 5. Top: $[P_{t_k/t_k}]_{11}$; Middle: $[P_{t_k/t_k}]_{22}$; Bottom: $[P_{t_k/t_k}]_{12}$.

Figure 8: realization of the observed equation (26) with $\lambda^2 = 600$ and $\sigma^2 = 100$.

Figure 9: first exact conditional moment together the estimated $\hat{x}_{t_k/t_k}$, given the observation of the Fig. 8.

Figure 10: second exact conditional moments together the estimated $P_{t_k/t_k}$, given the observation of the Fig. 8. Top: $[P_{t_k/t_k}]_{11}$; Middle: $[P_{t_k/t_k}]_{22}$; Bottom: $[P_{t_k/t_k}]_{12}$.
Figure 1:
Figure 2:
Figure 4:
Figure 5:
Figure 7:
Figure 8:
Figure 9: