Higher Order Local Linearization Methods for Solving Stochastic Differential Equations with additive noise

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Abstract

In this paper a new class of stable, high order methods for solving stochastic ordinary differential equations (SDEs) with additive noise is introduced. This is based on the addition of an stochastic correction term to the deterministic part of the LL approximation, which is determined as the solution of a suitable auxiliary SDE. For the latter, any discretization scheme can be used leading to what we call the High Order Stochastic Local Linearization Methods. In particular, combining with (explicit) Taylor or Runge Kutta methods yields to new classes of stable integrators. This permits to improve the order of convergence of the LL method without loss of its stability properties. The feasibility and performance of this methods are illustrated by means of computer simulations.

Key words: Local Linearization; stochastic differential equations; numerical solution; stability

1 Introduction

The present work introduces a new class of high order and stable numerical methods to compute the pathwise solution of stochastic differential equations (SDEs) with additive noise:

\[
dx(t) = f(t, x(t)) dt + \sum_{j=1}^{m} \int_{t_0}^{t} g^j(t) dw^j(t), \quad t \in [t_0, T] \quad x(t_0) = x_0, \tag{1}\]

where \(x(t) \in \mathbb{R}^d\), and \(f, g^j\) are smooth functions in such a way that the usual conditions for the existence and uniqueness of a path-wise strong solution of (1) are satisfied (for this conditions, see e.g., [1]) and \(w^j(t)\), \(j = 1, 2, \ldots, m\) are independent standard Wiener processes on a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\), whose increment \(\Delta w^j(t) = w^j(t + \Delta t) - w^j(t)\) is a Gaussian random variable \(\mathcal{N}(0, \Delta t)\).

In numerous fields and in a wide variety of situations, models of SDEs with additive noise of the above type (1) arises naturally in the modelling process, being used to express uncertainty and fluctuations, and serving also as possible basis for the investigation of qualitative behavior of dynamical systems, respect to the behavior under random perturbations which are state-independent. Examples of additive noise include blood clotting systems, cellular energetics and stochastic annealing; see chapter 5 in Kloeden and Platen [16], for further references. Since unfortunately analytic solution of the equations are rarely available, the use of numerical method to approximate their solutions are required.

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Up to now a plenty of methods have been developed (see extensive surveys in [16], [22]). In particular over the last years the Local Linearization methods (LL) for the solution of SDEs have been the subject of a number of papers (e.g., [5], [13], [14]). It consists, essentially in approximating the solution of the SDE in consideration by the explicit solution of the corresponding linear one resulting of replacing the drift term by its local linearization (first order deterministic Taylor in [5] or truncated Ito-Taylor in [13]) at each time step. Theoretical and simulations results indicate that the LL methods leads to efficient and stable schemes both for the approximated solution of SDEs as well as of ordinary differential equations (ODEs), the last in case of the diffusion coefficient been vanishes. This approach is consider among the most efficient numerical methods which are currently available. Nevertheless, the LL methods are of strong order of convergence at most 1.5 for SDEs, and order 2 for ODEs. This has motived the recent development of a class of higher order LL-based methods in the context of ODEs in [4] and posteriority in [10].

In this work we will focus in the construction of higher order stochastic LL methods, i.e., it can be thought of as an extension for SDEs of the mentioned approaches for ODEs. For this the class of methods we introduce here is based on the modification of the standard stochastic LL methods. It is based on the addition of an stochastic correction term to the deterministic part of the LL approximation, but now this is determined as the solution of an auxiliary SDE. For the latter, any discretization scheme can be used leading to what we call the High order Local Linearization method. In this way improvement to the strong order of convergence are achieved, while keeping the stability properties. In particular A-stability and nice asymptotical properties are obtained. This approach can be also thought of as a flexible framework for increasing the order of the LL solution as well as for stabilizing standard explicit integrators commonly used.

The structure of this paper is as follows. Section 2 reviews the standard order 1 and 1.5 LL methods. Section 3 introduces the class of High order LL methods, particularity two examples of different possible approaches are presented, as well as some implementation details are also discussed. Section 4 deals with the convergence of the schemes and its stability properties, it is proved that this approach makes possible to achieve an arbitrary order of convergence without losing important dynamic features of the standard LL method. In Section 5 is brevly discussed some implementation issues of the proposed schemes and finally Section 6 illustrates the performance of the introduced method through computer simulations.

## 2 Stochastic Standard Local Linearization methods

Let $(\tau)_h = \{t_0 < t_1 < \ldots < t_N = T\}$, be a given partition of $[t_0, T]$, and denote $h_n = t_{n+1} - t_n$, $\Lambda_n = [t_n, t_{n+1}]$, for $n = 0, \ldots, N - 1$.

The Stochastic Local Linearization (SLL) discretization $y_{t_h}^{LL}$ can be derived as follows (see, e.g., [11]). Once the initial value $y_{t_0}^{LL} = x_0$ has been specified, the approximations $y_{t_1}^{LL}, y_{t_2}^{LL}, \ldots, y_{t_N}^{LL}$ can be calculated recursively in the following way.

Define the local problems

$$
\mathbf{dx}(t) = \mathbf{f}(t, \mathbf{x}(t))dt + \mathbf{G}(t)d\mathbf{w}(t), \quad t \in \Lambda_n
$$

with initial condition $y_{t_n}^{LL}$ at $t = t_n$, where $\mathbf{G}(t)$ is the $d \times m$ matrix with columns $[\mathbf{G}(t)]^j = \mathbf{g}^j(t)$. These equations are approximated by linear ones on the basis of the first-order Taylor or
Ito-Taylor expansions around \((t_n, y_{tn}^{LL})\):

\[
dx(t) = \left( f'_x(t_n, y_{tn}^{LL})(x(t) - y_{tn}^{LL}) + a(t; t_n, y_{tn}^{LL}) \right) dt + G(t)dw(t) \tag{3}
\]

\[
x(t_n) = y_{tn}^{LL},
\]

where the function \(a\) is defined as

\[
a(t; t_n, y) = f(t_n, y) + f'_x(t_n, y)(t - t_n) \tag{4}
\]

for the Taylor expansion, and as

\[
a(t; t_n, y) = f(t_n, y) + \left( f'_x(t_n, y) + \frac{1}{2} \sum_{j=1}^{m} \left( I_{d \times d} \otimes \mathbf{g}^{(j)}(t_n) \right) f''(t_n, y) \mathbf{g}^{(j)}(t_n) \right) (t - t_n) \tag{5}
\]

for the Ito-Taylor expansion.

Here, \(f'_x\) and \(f'_y\) denote the partial derivatives of \(f\) with respect to the variables \(x\) and \(t\) respectively, and \(f''\) is the Hessian matrix of \(f\) at the point \((t_n, y)\), \(\otimes\) denotes the Kronecker product.

The problem (3) has an explicit solution \(y_{tn}^{LL}(., y_{tn}, t_n)\), see Arnold, [1], in terms of the fundamental matrix \(\Phi(t; t_n, y) = \exp \left( f'_x(t_n, y)(t - t_n) \right)\) of the corresponding deterministic homogeneous equation, i.e., \(\Phi(t; t_n, y)\) satisfies

\[
\begin{align*}
\Phi'(t; t_n, y_{tn}^{LL}) &= f'_x(t_n, y_{tn}^{LL})\Phi(t; t_n, y_{tn}^{LL}) \tag{6} \\
\Phi(t; t_n, y_{tn}^{LL}) &= I
\end{align*}
\]

Namely, for \(t \in \Lambda_n\),

\[
y_{tn}^{LL}(t, t_n, y_{tn}^{LL}) = y_{tn}^{LL} + \int_{t_n}^{t} \Phi(t; t_n, y_{tn}^{LL}) \Phi^{-1}(s; t_n, y_{tn}^{LL}) a(s; t_n, y_{tn}^{LL}) ds
\]

\[
+ \int_{t_n}^{t} \Phi(t; t_n, y_{tn}^{LL}) \Phi^{-1}(s; t_n, y_{tn}^{LL}) G(s) dw(s)
\]

Thus, \(y_{tn}^{LL}\) is obtained by

\[
y_{tn+1}^{LL} = y_{tn}^{LL}(t_{n+1}, t_n, y_{tn}^{LL})
\]

Summarizing the stochastic LL discretization can be expressed by the general recurrent relation:

\[
y_{tn+1}^{LL} = \varphi_n(t_{n+1}, y_{tn+1}^{LL}) + \xi_n(t_{n+1}, y_{tn+1}^{LL})
\]

starting at the point \(y_{t_0}^{LL} = x_0\), where

\[
\begin{align*}
\varphi_n(t, y) &= \varphi(t, t_n, y) = y + \int_{t_{n+1}}^{t} e^{(t-t_{n+1})i(t_n,y)} a(s + t_n, t_n, y) ds \tag{7} \\
\xi_n(t, y) &= y + \int_{t_{n+1}}^{t} e^{(t-s)i(t_n,y)} G(s) dw(s) \tag{8}
\end{align*}
\]

On the basis of these two discretizations various LL schemes have been proposed, which differ with respect to the way of computing the integrals in (7) and (8), see, e.g., [11]. Theoretically the convergence of the LL methods have been studied. Specifically in [11] it was proved that such discretization converge with strong global order \(\gamma = 1\), for \(a_n\) as in (4) and \(\gamma = 1.5\), for \(a\) as in (5).
3 High Order Stochastic Local Linearization Methods

In this section a new class of stochastic integrators for the approximation of the solution of (1) is introduced, this is derived by a modification of the SLL approach in order to improve accuracy and retaining desirable stability properties.

Let \( y_{tn}, \ t = 1, 2, \ldots, N \) said discretization, by starting at \( y_{t0} = x_0, \ \ y_{t+1} \) is updated from \( y_{tn} \) as follows.

From a particular case of the variation of constants formula in [21] it follows that the solution \( x_n(t) \) of the local problem (2) starting from \( y_{tn} \) at \( t = t_n \) can be written as

\[
x_n(t) = \varphi_n(t,y_{tn}) + r_n(t),
\]  

where

\[
r_n(t) = \Phi(t; t_n, y_{tn}) \left\{ \int_{t_n}^{t} \Phi_n^{-1}(t,t_n,y_{tn}) M(s,x_n(s); t_n,y_{tn}) ds + \int_{t_n}^{t} \Phi_n^{-1}(t,t_n,y_{tn}) G(s)dw(s) \right\},
\]

\[
M(s,x; t_n, y_{tn}) = f(s, x) - \left( f_2(t_n, y_{tn}) (x - y_{tn}) + a(s; t_n, y_{tn}) \right),
\]

and \( a \) is defined by (4) or (5). The following result provides an stochastic differential equation for \( r_n(t) \) in (10).

**Lemma 1** The function \( r_n(t) \) satisfies the, in general no autonoma, stochastic differential equation

\[
dr_n(t) = h(t, r_n(t)) dt + G(t)dw(t), \quad t \in \Lambda_n
\]

\[
r_n(t_n) = 0
\]

where

\[
h(t, r) = f'_2(t_n, y_{tn}) r + M(t, \varphi(t, t_n, y_{tn}) + r; t_n, y_{tn}), \quad t \in \Lambda_n
\]

**Proof.** For abbreviation here we set

\[
\varphi_n(t) = \varphi(t, t_n, y_{tn})
\]

\[
\Phi_n(t) = \Phi(t, t_n, y_{tn})
\]

\[
M_n(s, x) = M(s, x; t_n, y_{tn}),
\]

and denote by \( A_i \), and \( A^{ij} \) the \( i^{th} \) row and the \( (i, j) \)th element of the matrix \( A \) respectively. First note that \( r_n(t_n) = 0 \) is straightforward from (10). Setting

\[
u(t) = \int_{t_n}^{t} \Phi_n^{-1}(s) M(s, x_n(s)) ds + \int_{t_n}^{t} \Phi_n^{-1}(s) G(s)dw(s)
\]

then

\[
r_n(t) = \Phi_n(t) u(t)
\]

and hence

\[
x_n(t) = \varphi_n(t, y_{tn}) + \Phi_n(t) u(t),
\]
also \( u(t) \) has differential
\[
du(t) = e_t dt + F_t dw(t)
\]
where
\[
e_t = e(t, u(t)) = \Phi_n^{-1}(t) M_n(t, x_n(t)),
\]
\[
F_t = F(t) = \Phi_n^{-1}(t) G(t)
\]

Let \( r^k_n \) the \( k^{th} \) component of \( r_n(t) \), \( k = 1, 2, \ldots d \). By applying the multi-component Ito-Formula [16] to
\[
r^k_n = r^k_n(t, u^1, u^2, \ldots u^d) = \Phi_n^k u
\]
with \( u^\top = (u^1 u^2 \ldots u^d) \) then
\[
r^k_n(t) = \left\{ \frac{\partial r^k_n}{\partial t} + e_t^\top \nabla r^k_n + \frac{1}{2} tr \left( F_t F_t^\top \nabla [\nabla r^k_n] \right) \right\} dt + \nabla r^k_n F_t dw(t)
\]
where the partial derivatives are evaluated at \( (t, u(t)) \); \( \nabla \) is the gradient operator, \( ^\top \) the vector or matrix transpose operation. Thus \( \nabla r^k_n \) is the column vector of the first order spatial partial derivatives of \( r^k_n \).

First note that from (6) and (13)
\[
\frac{\partial r^k_n}{\partial t} = \frac{d\Phi_n^k}{dt} u = f^k_x(t_n, y_{1n}) \Phi_n(t) u = f^k_x(t_n, y_{1n}) r_n
\]
also note that since \( \nabla r^k_n = \Phi_n^k \) then
\[
e_t^\top \nabla r^k_n = (e_t^\top \nabla r^k_n)^\top = \Phi_n^k e_t,
\]
and
\[
\nabla r^k_n F_t = \Phi_n^k F_t
\]
on the other hand, obviously \( \frac{\partial^2 r^k_n}{\partial u^i \partial u^j} = 0 \), \( (i, j = 1, 2, \ldots d) \), hence
\[
\frac{1}{2} \text{tr} \left( F_t F_t^\top \nabla [\nabla r^k_n] \right) = \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^d \left( F_{t}^{i,j} F_{t}^{j,i} \frac{\partial^2 r^k_n}{\partial u^i \partial u^j} \right) = 0
\]
Thus for \( k = 1, 2, \ldots d \)
\[
dr^k_n(t) = \left\{ f^k_x(t_n, y_{1n}) r_n + \Phi_n e_t \right\} dt + \Phi_n^k F_t dw(t)
\]
Finally
\[
dr_n(t) = \left( \begin{array}{c}
-dr^1_n(t) \\
\vdots \\
dr^d_n(t)
\end{array} \right) = \left\{ f^1_x(t_n, y_{1n}) r_n(t) + \Phi_n(t) e_t \right\} dt + \Phi_n(t) F_t dw(t)
\]
\[
= \left\{ f^1_x(t_n, y_{1n}) r_n(t) + \Phi_n(t) \Phi_n^{-1}(t) M_n(t, \varphi_n(t) + r(t)) \right\} dt + \Phi_n(t) \Phi_n^{-1}(t) G(t) dw(t)
\]
\[
= \left\{ f^1_x(t_n, y_{1n}) r_n(t) + M_n(t, \varphi_n(t) + r(t)) \right\} dt + G(t) dw(t),
\]
consequently, the proof has been completed.

The previous lemma motives the contruction of the new class of High Order Stochastic Local Linearization methods, hereafter abbreviated HOSLL methods. For this, an approximation $z_n(t)$ to $r_n(t)$ is procured by solving (11) in each $\Lambda_n$ through one step of any extant numerical integrator. For this present paper the focus will only be on explicit methods, implicit methods will be consider in later works. Hence starting with $y_{t_0} = x_0$, a discretization can be obtained recursively by

$$y_{t_{n+1}} = \varphi(t_{n+1}; t_n, y_{t_n}) + z_n(t_{n+1})$$  \hspace{1cm} (14)

with the corresponding continuos-time approximation $y(t)$ defined by

$$y(t) = \varphi(t; t_n, y_{t_n}) + z_n(t), \quad t \in \Lambda_n$$  \hspace{1cm} (15)

where $y_{t_n}$ is the discretization (14).

Next, we will construct HOSLL schemes, based in the Stochastic Taylor and Runge Kutta Approaches. The presentation is restricted to order 2 methods, although this general methodology will allow the construction of high order methods in a systematic fashion as well. Also for simplicity assume constant step size $h$ and hereafter we will consider the function $a$ as in (4).

### 3.1 LLS-Taylor approach

Consider, the order 2 strong Taylor approximation $z_n(t)$ for (11). Because of the simplified nature of the stratonovich calculus, the stratonovich form of the stochastic Taylor series will be used here [16]. Thus at $t = t_{n+1}$

$$z_n(t_{n+1}) = z_n(t_n) + h \dot{w} + \frac{h^2}{2}L^0 + \sum_{j=1}^{m} g^j \Delta w^j + \sum_{j=1}^{m} L^j h J_{(j,0)} + \sum_{j=1}^{m} L^0 g^j J_{(0,j)}

+ \sum_{j_1, j_2 = 1}^{m} L^{j_1} L^{j_2} h J_{(j_1, j_2, 0)}$$

where the operators $L^j$ are defined as,

$$L^j f = (L^j f^1, \ldots, L^j f^d),$$

for $j = 0, \ldots, m$ and $f = (f^1, \ldots, f^d)$,

$$L^0 = \frac{\partial}{\partial t} + \sum_{i=1}^{m} f^i \frac{\partial}{\partial x^i},$$

$$L^j = \sum_{i=1}^{m} g^{ij} \frac{\partial}{\partial x^i}, \quad j = 1, \ldots, m$$

$J_{(j,0)}$, $J_{(0,j)}$, $J_{(j_1, j_2, 0)}$ represent Stratonovich multiple integrals and $\Delta w^j$, $j = 1, 2, \ldots, m$ are brownian increments of $w^j$.  

\hspace{1cm} 6
From the above relations and using the rules of the matrix differential calculus of [17]:

\[
\begin{align*}
\mathbf{h}(t_n, y_{tn}) &= 0 \\
L^0 \mathbf{h}(t_n, y_{tn}) &= 0 \\
L^1 \mathbf{h}(t_n, y_{tn}) &= f'_x(t_n, y_{tn}) g^j(t_n) \\
L^0 g^j(t_n, y_{tn}) &= g^j(t_n) \\
L^{j_1} L^{j_2} \mathbf{h}(t_n, y_{tn}) &= \left( I_{d \times d} \otimes g^{j_2 T}(t_n) \right) f''(t_n, y_{tn}) g^{j_1}(t_n)
\end{align*}
\]

Thus,

\[
\begin{align*}
\mathbf{z}_n(t_{n+1}) &= \sum_{j=1}^{m} g^j(t_n) \Delta \mathbf{w}^j + \sum_{j=1}^{m} f'_x(t_n, y_{tn}) g^j(t_n) \mathbf{J}_{(j,0)} + \sum_{j=1}^{m} g^j(t_n) \mathbf{J}_{(0,j)} \\
&\quad + \sum_{j_1, j_2=1}^{m} \left( I_{d \times d} \otimes g^{j_2 T}(t_n) \right) f''(t_n, y_{tn}) g^{j_1}(t_n) \mathbf{J}_{(j_1, j_2, 0)}
\end{align*}
\]

Hence the order 2 LLS_Taylor method is

\[
\begin{align*}
\mathbf{y}_{tn+1} &= \mathbf{v}_n(t_{n+1}; y_{tn}) + \sum_{j=1}^{m} g^j(t_n) \Delta \mathbf{w}^j + \sum_{j=1}^{m} f'_x(t_n, y_{tn}) g^j(t_n) \mathbf{J}_{(j,0)} + \sum_{j=1}^{m} g^j(t_n) \mathbf{J}_{(0,j)} \\
&\quad + \sum_{j_1, j_2=1}^{m} \left( I_{d \times d} \otimes g^{j_2 T}(t_n) \right) f''(t_n, y_{tn}) g^{j_1}(t_n) \mathbf{J}_{(j_1, j_2, 0)}
\end{align*}
\]

In this way we can construct practical codes from schemes derived of the above resulting approach, in situations where it is possible to compute efficiently the second derivatives of the drift term, i.e., por espcific dynamics always should be check whether there are considerable simplifications in this sense. However, in other cases, a posible disadvantages of this Taylor approach is that derivatives of various orders must be determinated. This can make the implementation of such schemes complicated task. Next one way to circumvent this difficult is considered.

### 3.2 LLS-Runge Kutta approach

Here a order 2 HOSLL scheme which avoid the use of upper derivatives is proposed, this is based in the well known class of Runge Kutta methods. For simplicity will be assumed \( m = 1 \), and the explicit order 2 strong RK approximation for additive noise from Kloeden and Platen, p. 384 in [16] is considered. It must be emphasized that any other Runge Kutta method from literature could be employed instead; e.g., the class of more recently RK methods of Burrage and Burrage [2], [3]. See also Schurz [22] for a complete review. Thus at \( t = t_{n+1} \):

\[
\mathbf{z}_n(t_{n+1}) = \frac{h}{2} \{ \mathbf{k}_1 + \mathbf{k}_2 \} + \mathbf{g}(t_n) \Delta \mathbf{w} + \frac{1}{h} \mathbf{J}_{(0,1)} \{ \mathbf{g}(t_{n+1}) - \mathbf{g}(t_n) \}
\]

where
\[ k_1 = h \left( t_n + \frac{1}{2} h, \gamma_+ \right) \]
\[ = f \left( t_n + \frac{h}{2}, \varphi_n \left( t_n + \frac{h}{2}, y_{t_n} \right) + \gamma_+ \right) - f'_x(t_n, y_{t_n}) \left( \varphi_n \left( t_n + \frac{h}{2}, y_{t_n} \right) - y_{t_n} \right) \]
\[ - \left( f(t_n, y_{t_n}) + f'_x(t_n, y) \frac{h}{2} \right) \]

\[ k_2 = h \left( t_n + \frac{1}{2} h, \gamma_- \right) \]
\[ = f \left( t_n + \frac{h}{2}, \varphi_n \left( t_n + \frac{h}{2}, y_{t_n} \right) + \gamma_- \right) - f'_x(t_n, y_{t_n}) \left( \varphi_n \left( t_n + \frac{h}{2}, y_{t_n} \right) - y_{t_n} \right) \]
\[ - \left( f(t_n, y_{t_n}) + f'_x(t_n, y) \frac{h}{2} \right) \]

with

\[ \gamma_\pm = z_n(t_n) + \frac{1}{2} h(t_n, z_n) h + \frac{1}{h_n} g(t_n) \left\{ J_{(1,0)} + \sqrt{2J_{(1,1,0)}h - \left( J_{(1,0)} \right)^2} \right\} \]
\[ = \frac{1}{h} g(t_n) \left\{ J_{(1,0)} + \sqrt{2J_{(1,1,0)}h - \left( J_{(1,0)} \right)^2} \right\} \]

Hence the order 2 LLS_RK methods is

\[ y_{t_{n+1}} = \varphi_n \left( t_{n+1}, y_{t_n} \right) + \frac{h}{2} \left\{ k_1 + k_2 \right\} \]
\[ + g(t_n) \Delta w + \frac{1}{h_n} \left\{ g(t_{n+1}) - g(t_n) \right\} J_{(0,1)} \]

(17)

4 Convergence and Stability

At first in this section, we will state the corresponding convergence theorem for the general LLSRK methods (14)-(15).

**Theorem 2** Let \( z(t) \) any numerical approximation for (11) with strong global order of convergence \( \gamma \), and \( h \ll 1 \) the discretization step size. Then the HOSLL approximation \( y(t) \) in (15) for the solution of (1) converges strongly and uniformly with the same order \( \gamma \), that is

\[ E \left( \sup_{t_0 \leq t \leq T} \| x(t) - y(t) \| \right) = O(h^\gamma) \]

To prove theorem (2) the following lemma concerning the order of strong convergence is used. This is a generalization, given by Schurz (see Chapter 5, theorem 5.6.5 in [22]), to the continuos time variant, of the lemma on the order of convergence in Milstein [18],
Lemma 3 Suppose that the one-step discrete time approximation $Y(t)$, approximating $X(t)$, has order of accuracy $\gamma_1$ and $\gamma_2$ for local mean error and mean-square error respectively; more precisely, for arbitrary $t_0 \leq t \leq T - h$, $x \in \mathbb{R}$ the following inequalities hold:

$$\|E(X_{t,z}(t+h) - Y_{t,z}(t+h) \mid F_t)\| \leq K_1 \left(1 + \|z\|^2\right)^{\frac{1}{2}} h^{\gamma_1}$$  \hfill (18)

$$\left[E\left(\|X_{t,z}(t+h) - Y_{t,z}(t+h)\|^2 \mid F_t\right)\right]^{\frac{1}{2}} \leq K_2 \left(1 + \|z\|^2\right)^{\frac{1}{2}} h^{\gamma_2}$$  \hfill (19)

with $\gamma_2 \geq \frac{3}{4}$ and $\gamma_1 \geq \gamma_2 + \frac{1}{2}$ then, the Fundamental Strong Mean Square Convergence Relation holds:

$$\left(E\left(\sup_{t_0 \leq t \leq T} \|X_{t_0,x_0}(t) - Y_{t_0,x_0}(t)\|^2 \mid F_{t_0}\right)\right)^{\frac{1}{2}} \leq K \left(1 + \|x_0\|^4\right)^{\frac{1}{2}} h^{\gamma_2 + \frac{1}{2}}$$  \hfill (20)

Proof of Theorem (2). This is an straightforward result from the equality:

$$x_{t,z}(t+h) - y_{t,z}(t+h) = (\varphi(t+h;t,z) + r(t+h)) - (\varphi(t+h;t,z) + z(t+h)) = r(t+h) - z(t+h)$$

In fact, since $z$ has strong global order of convergence $\gamma$ approximating $r$, the relations (18)-(19) holds for $X = z$ and $Y = r$, with $\gamma_2 = \gamma$, so (20) is satisfied by $X(t) = x(t)$ and $Y(t) = y(t)$.

Next the A-Stability of the HOSLL methods will be discussed. So, following section 8 of chapter 9 in [16], we shall choose the class of complex-valued linear test equation

$$dX(t) = \lambda X(t) + dW(t)$$  \hfill (21)

where the parameter $\lambda$ is a complex number with Re $\{\lambda\} < 0$ and $W$ is a real valued standard Wiener process. This equation has an ergodic solution (for Re $\{\lambda\} < 0$), which makes it a good choice. Under the assumption that a given scheme, with step size $h$, applied to (21) allows a representation in the form

$$y_{n+1} = \mathbb{R}(\lambda h)y_n + Z_n$$

where $\mathbb{R}$ is a complex value function and $Z_n$ represent random variables with do not depend on $\lambda$ or $y_0, y_1, \ldots, y_{n+1}$, similar as in the deterministic scenario, the set of complex numbers

$$S = \{\lambda h \in \mathbb{C} : \text{Re} \{\lambda\} < 0, \quad |\mathbb{R}(\lambda h)| < 1\}$$

is called the region of absolute stability of the scheme. A stochastic scheme is then called A-stable if its region of absolute stability contains the whole left half of the complex plane. The regions of absolute stability for some numerical schemes are considered in ([16]).

**Theorem 4** The HOSLL methods (16) and (17) are A-stables

Proof. By Applying the method (16) and (17) to the test problem (21), note that

$$\varphi_n(t_{n+1}, y_{t_n}) = y_{t_n} + \int_0^h e^{(h-s)}\lambda a(s + t_n; t_n, y_{t_n})ds = \mathbb{R}(\lambda h)$$

where $\mathbb{R}(\lambda h)$ is regions of absolute stability of the deterministic LL method, see [12]. On the other hand it is no hard to see that for both methods, $Z_n = \Delta w + \lambda J_{(1)}$. Hence the A-stability of this methods is inherited from the A-stability of the deterministic LL method. ■
5 Implementations issues

When implementing the methods (16) and (17) and other resulting HOLLS methods, to generate the Statonovich integrals $\mathbf{J}_{(j,0)}$, $\mathbf{J}_{(0,j)}$, $\mathbf{J}_{(j,j,0)}$ and the increment $\Delta \mathbf{w}^j$ is needed. They can be approximated, for instance, based on Fourier series expansions, as shown in chapter 5 of [16]. For completeness and simplicity in the exposition here we will present such methods for obtaining, in each integration interval, the approximations $\Delta \mathbf{w}^p$, $\mathbf{J}_{(1,0)}^p$, $\mathbf{J}_{(0,1)}^p$, $\mathbf{J}_{(1,1,0)}^p$, to $\Delta \mathbf{w}$, $\mathbf{J}_{(1,0)}$, $\mathbf{J}_{(0,1)}$, $\mathbf{J}_{(1,1,0)}$, with truncation index $p$, which for $p = 5$, $p = 15$ or $p = 20$, has been considered as appropriated (see e.g. [3]).

For $r = 1, \ldots, p$, we shall define independent standard Gaussian random variables $\zeta_1$, $\xi_{1,r}$, $\eta_{1,r}$, $\mu_{1,r}$, $\theta_{1,r}$, and using these random variables the above approximations are

$$\Delta \mathbf{w}^p = \sqrt{h} \zeta_1$$
$$\mathbf{J}_{(1,0)}^p = \frac{h}{2} (\Delta \mathbf{w}^p + a_{1,0})$$
$$\mathbf{J}_{(0,1)}^p = \frac{h}{2} (\Delta \mathbf{w}^p - a_{1,0})$$
$$\mathbf{J}_{(1,1,0)}^p = \frac{h^2}{6} \zeta_1^2 + \frac{h}{4} a_{1,0}^2 - \frac{1}{2\pi} h \xi_{1,b} b_1 + \frac{3}{4} a_{1,0} \zeta_1 - h^2 C_{1,1}^p$$

with

$$a_{1,0} = -\frac{1}{\pi} \sqrt{2h} \sum_{r=1}^{p} r \xi_{1,r} - 2 \sqrt{h} \rho_p \mu_{1,r}$$
$$\rho_p = \frac{1}{12} - \frac{1}{2\pi^2} \sum_{r=1}^{p} \frac{1}{r^2}$$
$$b_1 = \sqrt{\frac{h}{2}} \sum_{r=1}^{p} \frac{1}{r^2} \eta_{1,r} + \sqrt{\alpha_p} \theta_{1,r}$$
$$\alpha_p = \frac{\pi^2}{180} - \frac{1}{2\pi^2} \sum_{r=1}^{p} \frac{1}{r^4}$$
$$C_{1,1}^p = -\frac{1}{2\pi^2} \sum_{r,l=1}^{p} \frac{r}{r^2 - l^2} \left( \frac{1}{l^2} \xi_{1,r} \xi_{1,l} - \frac{1}{r^2} \eta_{1,r} \eta_{1,l} \right)$$

On the other hand, in reference with $\varphi_n(t, y)$ in (7). It can be implementing through

$$\varphi_n(t, y) = y + \mathbf{v}_n(t)$$

where from the theorem 1 in [23] it follows that the vector $\mathbf{v}_n(t)$ can be obtained from the identity in the block matrix

$$\begin{bmatrix}
\mathbf{F}(t; t_n, y) & \mathbf{b}_1(t; t_n, y) & \mathbf{v}_n(t) \\
0 & 1 & \mathbf{b}_2(t; t_n, y)
\end{bmatrix} = e^{(t-t_n)\mathbf{C}_n}$$

10
where 
\[
C_n = \begin{bmatrix}
  f'_x(t_n, y) & f'_y(t_n, y) & f(t_n, y) \\
  0 & 0 & 1 \\
  0 & 0 & 0
\end{bmatrix} \in \mathbb{R}^{(d+2) \times (d+2)}
\]
and \(F(t; t_n, y) \in \mathbb{R}^{d \times d}, b_1(t; t_n, y) \in \mathbb{R}^d, b_2(t; t_n, y) \in \mathbb{R}\) are certain vectors.

It is also worth of noting some points that contribute to decrease the burden of computation. First from the fact that \(h(t_n, 0) = 0\) and its implications for instance in the LLS-Taylor approach. Second in noting that the computation of just one matrix exponential \(\exp(ah_nC_n)\), and a few powers of this exponential per step is just required, where \(C_n\) is certain matrix, for the LLS-Runge Kutta approach. The last consideration is remarkable when choosing the mentioned approach based in the class of Runge Kutta methods of Burrage and Burrage.

6 Numerical examples

In this section numerical results are reported to confirm the performance of the HOSLL methods (16) and (17) through some computer simulations. In the examples a comparation among the Runge Kutta methods of Kloeden, the Local Linearization methods and the HOLL methods is achieved.

The first equation considered is a well known nonlinear oscillator; the stochastic Van der Pol random oscillator, which is formulated:

\[
\begin{align*}
x'_1 &= x_2, \\
x'_2 &= E \left( (1 - x_1^2) x_2 - x_1 \right) + \sigma d\mathbf{w},
\end{align*}
\]

with \(E = 10\) and we will consider how work the different methods when the noises \(\sigma\) increase. Figures 1, 2, 3, show the numerical solution when \(\sigma\) takes the values \(\sigma_1 = 200, \sigma_2 = 400, \sigma_3 = 600\) respectively. Note that even for moderate step size the HOLL methods replicates much better the actual dynamics of the systems than the other methods, which shows a tend to explosive trajectories when noise change increasing. Even for \(\sigma = 600\) with the same step size the HOLL method shows adequated accuracy and reproduce correctly the phase portrait in contrast with the rest of the methods. This demonstrates that the new methods can in practice leads to improvement in dynamical performance.

The second example is also a nonlinear problem, namely the stochastic planar Brusselator, which is commonly used for modeling unforced periodic oscillations in certain chemical reactions, but in this case pertubated additively:

\[
\begin{align*}
x'_1 &= a - (b + 1) x_1 + x_1^2 x_2 + \sigma d\mathbf{w}, \\
x'_2 &= bx_1 - x_1^2 x_2,
\end{align*}
\]

where \(a = 1, b = 3,\)

It is known that for this parameters the trajectory shows a limit cicle in case that ..., note in Figure 4, and Figure 5, that the LLSRK method has satisfactory performance for the differents noises \(\sigma_1 = 35, \sigma_2 = 70\) in comparation with the other methods which fails when noise increase,....

In the following table, we report the relative CPU times of the differents methods when solving the Van der Pol equation with \(\sigma = 400\). This shows that the proposed method in this
paper not only preserve the qualitative properties of the stochastic equations, but also a low CPU time when comparing with the rest of the methods:

<table>
<thead>
<tr>
<th>Scheme</th>
<th>RK</th>
<th>LLRK</th>
<th>LL</th>
<th>IRK</th>
</tr>
</thead>
<tbody>
<tr>
<td>Relative Time</td>
<td>1</td>
<td>1.27</td>
<td>1.87</td>
<td>19.58</td>
</tr>
</tbody>
</table>

References

Figure 1

- Taylor 2
- SRK2 Burrage
- SRK2 Kloeden

Graphs for different step sizes:
- $h = 2 \times 10^{-6}$
- $h = 2 \times 10^{-7}$
- $h = 2 \times 10^{-11}$
Figure 2
Figure 3

Three plots are shown, each with a different step size $h$.

1. $h = 2 \times 10^{-6}$
2. $h = 2 \times 10^{-7}$
3. $h = 2 \times 10^{-11}$

The plots compare the LLSRK2 and Taylor 2 methods.