Linear estimation of continuous-discrete linear state space models with multiplicative noise

J.C. Jimenez\textsuperscript{a},\textsuperscript{*}, T. Ozakib\textsuperscript{b}

\textsuperscript{a}Instituto de Cibernética, Matemática y Física, Calle 15, e/ C y D, Vedado, La Habana 4, C.P. 10400, Cuba
\textsuperscript{b}Depto. of Prediction and Control, The Institute of Statistical Mathematics, 4-6-7 Minami-Azabu, Minato-ku, Tokyo 106-8569, Japan

Received 23 October 2000; received in revised form 17 December 2001; accepted 15 March 2002

Abstract

This paper deals with the estimation of the state variable of continuous-discrete linear state space models with multiplicative noise. Specifically, the optimal minimum variance linear filter for that class of models is constructed. Moreover, the solutions of the differential equations that describe the evolution of the two first conditional moments between observations are obtained and an algorithm for their numerical computation is also given. The performance of the linear filter is illustrated by means of numerical experiments. © 2002 Elsevier Science B.V. All rights reserved.

Keywords: Optimal minimum variance estimation; Linear state space models; Multiplicative noise; Kalman filter

1. Introduction

The estimation of the state of a continuous stochastic dynamical system from noisy discrete observations taken on the state is of central importance to solve diverse scientific and technological problems. The major contribution to the solution of this estimation problem is due to Kalman [10–12], who provided a sequential and computationally efficient solution to the optimal filtering and prediction problem for linear state space models with additive noise.

Unfortunately, this Kalman algorithm is restricted to the class of continuous linear systems with additive noise. So that, it is not applicable to the large variety of estimation problems that involve linear stochastic differential equations with multiplicative noise. It includes important problems in chemistry, biology, ecology, economics, physics and engineering (see [13,15] and references therein).

It is well known that, for linear models with multiplicative noise, the state is a non-Gaussian process [18] and that the optimal filter is nonlinear and very difficult to compute in practice [14]. This is the reason for which, for this kind of models, the construction of high-quality suboptimal filters that approximate the optimal one becomes very important. Prominent examples of suboptimal filters are the linear ones, which have been widely used for the estimation of the state of both, continuous–continuous [14,16,18] and discrete–discrete [3,17,18] models. Recently, quadratic [4] and polynomial suboptimal filters [2] have also been proposed for the discrete–discrete models. However, no suboptimal filters for continuous-discrete models with multiplicative noise have been considered.

* Corresponding author.
The proposal of this paper is to construct an optimal linear filter for this class of models.

The paper is organized as follows. In Section 2, the optimal minimum variance linear filter is introduced. In Section 3, the solutions of the differential equations that describe the evolution of the first two conditional moments between observations are obtained and, in Section 4, an algorithm for their numerical computation is presented. In the last section, the effectiveness of the filter is illustrated by means of numerical experiments.

2. Optimal minimum variance linear filter

Let the linear state space model defined by the continuous state equation
\[
dx(t) = (A(t)x(t) + a(t))\,dt
\]
\[+ \sum_{i=1}^{m}(B_i(t)x(t) + b_i(t))\,dw_i(t)
\] (1)
and the discrete observation equation
\[
z_t = C(t)x(t) + \sum_{i=1}^{n}D_i(t)x(t)\xi_{ti}^i + F(t)\epsilon_t
\] for \(k = 0, 1, \ldots, N\), (2)
where \(x(t) \in \mathbb{R}^d\) is the state vector at the instant of time \(t\), \(z_t \in \mathbb{R}^r\) is the measurement vector at the instant of time \(t\), \(w\) is an \(m\)-dimensional Wiener process with independent components, and \(\{\xi_{ti}^i \sim \mathcal{N}(0, \Lambda), \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)\}\) and \(\{\epsilon_t \sim \mathcal{N}(0, \Sigma)\}\) are sequences of random vectors i.i.d.

Let \(\hat{x}_{t/\rho} = \mathbb{E}(x(t)/Z_\rho)\) and \(P_{t/\rho} = \mathbb{E}((x(t) - \hat{x}_{t/\rho})(x(t) - \hat{x}_{t/\rho})^T/Z_\rho)\) for all \(\rho \leq t\), where \(\mathbb{E}(\cdot)\) denotes expected value and \(Z_\rho = \{z_0 : t_k \leq \rho\}\).

Suppose that \(\mathbb{E}(w(t)w^T(t)) = I, \mathbb{E}(\xi_{ti}^i\epsilon_t) = \theta_i(t_k), \hat{x}_{0/0} = \mathbb{E}(x(t_0)/Z_0) < \infty\) and \(P_{0/0} = \mathbb{E}((x(t_0) - \hat{x}_{0/0})(x(t_0) - \hat{x}_{0/0})^T/Z_{0/0}) < \infty\).

**Theorem 1.** The optimal (minimum variance) linear filter for the linear models (1)–(2) consists of equations of evolution for the conditional mean \(\hat{x}_{t/\rho}\) and covariance matrix \(P_{t/\rho}\). Between observations, these satisfy the ordinary differential equations
\[
d\hat{x}_{t/\rho} = (A(t)\hat{x}_{t/\rho} + a(t))\,dt
\] (3) and
\[
dP_{t/\rho} = \left\{ A(t)P_{t/\rho} + P_{t/\rho}A^T(t) + \sum_{i=1}^{m}b_i(t)b_i^T(t)
\right. \]
\[+ \sum_{i=1}^{m}B_i(t)(P_{t/\rho} + \hat{x}_{t/\rho}\hat{x}_{t/\rho}^T)B_i^T(t)
\]
\[+ \sum_{i=1}^{m}(B_i(t)\hat{e}_i + b_i(t)\hat{e}_i^T)B_i^T(t)) \right\} dt
\] (4)
for all \(t \in [t_k, t_{k+1})\). At an observation at \(t_k\), they satisfy the difference equation
\[
\hat{x}_{t_k+1/\rho} = \hat{x}_{t_k+1/\rho} + K_{t_k+1} (z_{t_k+1} - C(t_{k+1})\hat{x}_{t_k+1/\rho}),
\] (5)
\[
P_{t_k+1/\rho} = P_{t_k+1/\rho} - K_{t_k+1} C(t_{k+1}) P_{t_k+1/\rho},
\] (6)
where
\[
K_{t_k+1} = P_{t_k+1/\rho} C^T(t_{k+1})(C(t_{k+1})P_{t_k+1/\rho}C^T(t_{k+1})
\]
\[+ F(t_{k+1})\Sigma F^T(t_{k+1}) + \sum_{i=1}^{n} \lambda_i D_i(t_{k+1})
\]
\[\times (P_{t_k+1/\rho} + \hat{x}_{t_k+1/\rho}\hat{x}_{t_k+1/\rho}^T)D_i^T(t_{k+1})
\]
\[+ \sum_{i=1}^{n} D_i(t_{k+1})\hat{x}_{t_k+1/\rho}(\theta_i(t_{k+1}))^T F(t_{k+1})
\]
\[+ \sum_{i=1}^{n} F(t_{k+1})\theta_i(t_{k+1})\hat{x}_{t_k+1/\rho}^{-1} D_i^T(t_{k+1})^{-1}
\] (7)
is the filter gain. The predictions \(\hat{x}_{t/\rho}\) and \(P_{t/\rho}\) are accomplished, respectively, via expressions (3) and (4) with initial conditions \(\hat{x}_{0/\rho}\) and \(P_{0/\rho}\) and \(\rho < t\).

**Proof.** Obviously, in absence of observations (i.e., between observations) one has that \(\mathbb{E}(x(t)) = \mathbb{E}(x(t)/Z_t) = \mathbb{E}(x(t)/Z_{t-})\), for all \(t \in [t_k, t_{k+1})\) and \(\tau < t\). Moreover, it is well known that in this case the conditional mean is the minimum variance estimate for the filtering and prediction problem [7].

Eq. (3) for the conditional mean follows taking expectation on both sides of the integral form of (1).
By definition the differential of the conditional covariance matrix is
\[ dP_{i,j} = d\left(\mathcal{E}\left(x_i x_j^T/Z_t\right)\right) - d\left(\hat{x}_i\hat{x}_j^T\right). \] (8)

Using the Ito formula [13] it is obtained that
\[ d\left(\mathcal{E}\left(x_i x_j^T/Z_t\right)\right) = \{A(t)\mathcal{E}\left(x_i x_j^T/Z_t\right) + \mathcal{E}\left(x_i x_j^T/Z_t\right)A^T(t)\} dt + \sum_{i=1}^{m} B_i(t)\mathcal{E}\left(x_i x_j^T/Z_t\right)B_i^T(t) dt + \sum_{i=1}^{m} \left(B_i(t)\hat{x}_i\hat{b}_i^T(t) + b_i(t)\hat{x}_i^T B_i^T(t) + b_i(t)b_i^T(t)\right) dt \] (9)

and from expression (3) it is obtained that
\[ d(\hat{x}_i\hat{x}_j^T) = (A(t)\hat{x}_i\hat{x}_j^T + \hat{x}_i\hat{x}_j^TA^T(t) dt + \sum_{i=1}^{m} B_i(t)\hat{x}_i\hat{b}_i^T(t) dt + \sum_{i=1}^{m} (B_i(t)\hat{x}_i\hat{b}_i(t) + b_i(t)\hat{x}_i^T B_i^T(t) + b_i(t)b_i^T(t)) dt. \] (10)

Thus, expression (4) follows from substituting (9) and (10) in (8).

At an observation at \( t_{k+1} \), let us consider the following unbiased filter estimate:
\[ \hat{x}_{k+1} = \hat{x}_{k+1} + K_{k+1}(z_{k+1} - C(t_{k+1})\hat{x}_{k+1}), \] (11)

where \( K_{k+1} \) is the matrix that minimizes the functional
\[ \mathcal{E}(\hat{x}_{k+1}^T U \hat{x}_{k+1}), \] (12)

where \( \hat{x}_{k+1} = x_{k+1} - \hat{x}_{k+1}/u_{k+1} \) and \( U \) is a symmetric positive matrix.

By substituting (2) in (11) it is obtained that
\[ \mathcal{E}(\hat{x}_{k+1}^T \hat{x}_{k+1}) = (I - K_{k+1} C(t_{k+1}))\mathcal{E}(P_{k+1}/u_{k+1})(I - K_{k+1} C(t_{k+1}))^T + \sum_{i=1}^{n} \lambda_i K_{k+1} D_i(t_{k+1})\mathcal{E}(P_{k+1}/u_{k+1}) \]
\[ + \hat{x}_{k+1}^T \hat{x}_{k+1} D_i^T(t_{k+1})K_{k+1}^T + \sum_{i=1}^{n} K_{k+1} D_i(t_{k+1})\mathcal{E}(\hat{x}_{k+1}/u_{k+1}) \]
\[ \times (\theta'(t_{k+1}))^T F^T(t_{k+1}) K_{k+1}^T + \sum_{i=1}^{n} K_{k+1} F(t_{k+1}) \theta'(t_{k+1})\mathcal{E}(\hat{x}_{k+1}/u_{k+1}) \]
\[ \times D_i^T(t_{k+1}) K_{k+1}^T + K_{k+1} F(t_{k+1})\Sigma F^T(t_{k+1}) K_{k+1}^T. \] (13)

Using that \( \mathcal{E}(\hat{x}_{k+1}^T U \hat{x}_{k+1}) = \text{tr}(\mathcal{E}(\hat{x}_{k+1}^T \hat{x}_{k+1}) U) \) it follows that
\[ \partial \mathcal{E}(\hat{x}_{k+1}^T U \hat{x}_{k+1}) / \partial K_{k+1} = 2UK_{k+1} \{ C(t_{k+1})\mathcal{E}(P_{k+1}/u_{k+1})C^T(t_{k+1}) \]
\[ + F(t_{k+1})\Sigma F^T(t_{k+1}) \]
\[ + \sum_{i=1}^{n} \lambda_i D_i(t_{k+1})\mathcal{E}(P_{k+1}/u_{k+1}) \]
\[ + \hat{x}_{k+1}/u_{k+1} D_i^T(t_{k+1}) \]
\[ + \sum_{i=1}^{n} F(t_{k+1}) \theta'(t_{k+1})\mathcal{E}(\hat{x}_{k+1}^T \hat{x}_{k+1}) \]
\[ - 2U \mathcal{E}(\hat{x}_{k+1}^T \hat{x}_{k+1}) C^T(t_{k+1}). \]

Taking into account some trivial properties of the expectation operator \( \mathcal{E} \) and that \( \mathcal{E}(P_{k+1}/u_{k+1}) \neq 0 \) then, expression (7) for \( K_{k+1} \) is finally obtained by solving
\[ \partial \mathcal{E}(\hat{x}_{k+1}^T U \hat{x}_{k+1}) / \partial K_{k+1} = 0. \]

The expression (6) is obtained by substituting (7) in (13) and using that \( \mathcal{E}(\hat{x}_{k+1}^T \hat{x}_{k+1}) = \mathcal{E}(P_{k+1}/u_{k+1}) \). \( \Box \)

Note that, for linear models (1)–(2) with \( a(t) \equiv 0 \) and \( B_i(t) \equiv D_i(t) \equiv 0 \) (for all \( i = 1, \ldots, m \) and \( j = 1, \ldots, n \)), filters (3)–(6) reduces to the (optimal minimum variance) Kalman–Busby filter for continuous-discrete models with additive noise [10–12].

3. Solution of the equations for the first two conditional moments between observations

This section deals with the problem of solving the ordinary differential equations that appear in
Theorem 1. Specifically, the solution of the equations for the predictions \( \hat{x}_{t/t_k} \) and \( P_{t/t_k} \) will be given.

Here, the symbols vec, \( \oplus \) and \( \otimes \) will denote the vectorization operator, the Kronecker sum and product, respectively.

**Theorem 2.** The general solution of the system of differential equations (3)–(4) is given by

\[
\hat{x}_{t/t_k} = \exp \left( \int_0^{t-t_k} A(u + t_k) \, du \right) \hat{x}_{t_k/t_k} \\
+ \int_0^{t-t_k} \exp \left( - \int_0^s A(u + t_k) \, du \right) \, a(s + t_k) \, ds
\]

and

\[
\text{vec}(P_{t/t_k}) = \exp \left( \int_0^{t-t_k} \mathcal{A}(u + t_k) \, du \right) \text{vec}(P_{t_k/t_k}) \\
+ \int_0^{t-t_k} \exp \left( - \int_0^s \mathcal{A}(u + t_k) \, du \right) \\
\times \mathcal{B}(s + t_k) \, ds,
\]

for all \( t \in [t_k, t_{k+1}) \), where

\[ \mathcal{A}(s) = A(s) \oplus A(s) + \sum_{i=1}^m B_i(s) \otimes B_i(s) \]

and

\[ \mathcal{B}(s) = \sum_{i=1}^m (B_i(s) \otimes B_i(s)) \text{vec}(\hat{x}_{s/t_k})^T \]

+ \sum_{i=1}^m \text{vec}(b_i(s) \otimes b_i(s)) + \sum_{i=1}^m (b_i(s) \otimes B_i(s))

+ B(s) \otimes b(s) \text{vec}(\hat{x}_{s/t_k}).

In particular, if \( A(t) = A \), \( B(t) = B \), \( a(t) = a \) and \( b(t) = b \) for all \( t \), then the above solution reduces to

\[
\hat{x}_{t/t_k} = \hat{x}_{t_k/t_k} + \int_0^{t-t_k} \exp(\mathcal{A}s) \, ds (\mathcal{A} \hat{x}_{s/t_k} + a)
\]

and

\[
\text{vec}(P_{t/t_k}) = \exp(\mathcal{A}(t - t_k)) \left( \text{vec}(P_{t_k/t_k}) \\
+ \sum_{i=1}^s \int_0^{t-t_i} \exp(-\mathcal{A}s) \mathcal{B}_i \exp(\mathcal{A}s) \mathcal{C}_i \, ds \right),
\]

where \( \mathcal{A} = A \oplus A + \sum_{i=1}^m B_i \otimes B_i \), and \( \mathcal{A}_i \), \( \mathcal{B}_i \) and \( \mathcal{C}_i \) are the matrices defined in Table 1.

The following lemma will be useful to demonstrate the above theorem.

**Lemma 3** (Van Loan [20]). Let \( A_1 \), \( A_2 \) and \( A_3 \) be square matrices, \( n_1 \), \( n_2 \) and \( n_3 \) be positive integers, and set \( m \) to be their sum. If the \( m \times m \) block triangular matrix \( C \) is defined by

\[
C = \begin{bmatrix} A_1 & B_1 & C_1 \\ 0 & A_2 & B_2 \\ 0 & 0 & A_3 \end{bmatrix}
\]

then, for \( s \geq 0 \)

\[
\begin{bmatrix} F_1(s) & G_1(s) & H_1(s) \\ 0 & F_2(s) & G_2(s) \\ 0 & 0 & F_3(s) \end{bmatrix} = \exp(sC),
\]

where

\[
F_j(s) \equiv \exp(A_j s) \quad \text{for } j = 1, 2, 3,
\]

\[
G_j(s) \equiv \int_0^s \exp(A_j(s - u))B_j \exp(A_{j+1}u) \, du
\]

for \( j = 1, 2, 3 \),

\[
H_1(s) \equiv \int_0^s \exp(A_1(s - u))C_1 \exp(A_3u) \, du
\]

+ \int_0^s \int_0^u \exp(A_1(s - u))B_1 \exp(A_2(u - r)) \\
\times B_2 \exp(A_3 r) \, dr \, du.
\]

**Proof of Theorem 2.** Solution (14) of the linear equation (3) is very well known [1]. Likewise, (15) is also solution of a linear equation. Specifically, it is the solution of

\[
d \text{vec}(P_{t/t_k}) = (\mathcal{A}(t) + \mathcal{A}(t) \text{vec}(P_{t/t_k})) \, dt.
\]
which is obtained by applying the operator vec to Eq. (4).

In the case that \( A(t) = A, B(t) = B, a(t) = a \) and \( b(t) = b \) for all \( t \), expressions (14) and (15) reduce to

\[
\hat{x}_{tk} = \exp(A(t - t_k)) \left( x_{tk} + \int_{0}^{t - t_k} \exp(-A\tau) a \, d\tau \right) \tag{20}
\]

and

\[
\text{vec}(P_{tk}) = \exp(\mathcal{A}(t - t_k))(\text{vec}(P_{tk})) \\
+ \int_{0}^{t - t_k} \exp(-\mathcal{A}s) \mathcal{B}(s + t_k) \, ds, \tag{21}
\]

respectively.

Expression (16) is obtained from (20) and the identity \( \int_{0}^{t - t_k} \exp(-A\tau) \, d\tau A = -(\exp(-A(t - t_k)) - I) \).

Using Lemma 3, expression (16) can be rewritten as

\[
\hat{x}_{tk} = \hat{x}_{tk} + L^T \exp((t - t_k)C)R, \tag{22}
\]

where

\[
C = \begin{bmatrix} A & A\hat{x}_{tk} + a \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{(d+1) \times (d+1)}
\]

and \( L^T = [I_d \ 0_{d \times 1}] \) and \( R^T = [0_{1 \times d}] \). Then, substituting (22) in (21), and after some algebraic manipulations expression (17) is obtained. \( \square \)

4. Numerical computation of the prediction estimates

In the previous section, expressions for the prediction \( \hat{x}_{tk} \) and \( P_{tk} \) were obtained. However, in general, these expressions are difficult to implement in practice by means of a computer program. In this section, simple algebraic expressions are introduced as a computationally feasible alternative to (16) and (17).

**Theorem 3.** Expressions (16) and (17) are equivalent to

\[
\hat{x}_{tk} = \hat{x}_{tk} + g(t) \tag{23}
\]

and

\[
\text{vec}(P_{tk}) = \mathcal{F}_1(t) \text{vec}(P_{tk}) + \sum_{i=1}^{5} \mathcal{G}_i(t) \mathcal{G}_i, \tag{24}
\]

respectively. Here, the vector \( g(t) \) and the matrices \( \mathcal{F}_1(t) \) and \( \mathcal{G}_i(t) \) are defined by matrix identities

\[
\begin{bmatrix} F(t) & g(t) \\ 0 & 1 \end{bmatrix} = \exp((t - t_k)C),
\]

\[
\begin{bmatrix} \mathcal{F}_i(t) & \mathcal{G}_i(t) \\ 0 & \mathcal{F}_i(t) \end{bmatrix} = \exp((t - t_k)\mathcal{G}_i),
\]

where

\[
C = \begin{bmatrix} A & A\hat{x}_{tk} + a \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{(d+1) \times (d+1)},
\]
$J_i = \begin{bmatrix} \mathcal{A} & \mathcal{B}_i \\ 0 & \mathcal{A}_i \end{bmatrix}$

and the matrices $\mathcal{A}$, $\mathcal{A}_i$, $\mathcal{B}_i$, and $\mathcal{C}_i$ are defined as in Theorem 2.

Proof. Expression (23) is straightly derived rewriting (16) as

$$\hat{x}_{t_k/h} = \hat{x}_{t_k/h} + \int_0^{t_k} \exp(A(t - t_k - u))du(A\hat{x}_{t_k/h} + a)$$

and using Lemma 3, with $s = t - t_k$, $A_1 = A$, $B_1 = A\hat{x}_{t_k/h} + a$, and $A_2 = 0$ in (18).

Expression (24) is obtained by using Lemma 3, with $s = t - t_k$, $A_1 = \mathcal{A}$, $B_1 = \mathcal{B}_i$, and $A_2 = \mathcal{A}_i$ in (18). Obviously, $\mathcal{F}(t_k) = \exp((t - t_k)\mathcal{A})$. □

In this way, the numerical computation of the conditional moments $\hat{x}_{t_k/h}$ and $P_{t_k/h}$ is reduced to use an convenient algorithm to compute matrix exponentials, e.g., those based on rational Padé approximations [5], the Schur decomposition [5] or Krylov sub-space methods [6] (for a recent review see [19]). The selection of one of them will mainly depend of the size and structure of the matrix $A$. In many cases, it is enough to use the algorithm developed in [20] which takes advantage of the special structure of the matrices $C$ and $J_i$. However, for state space models that involve more than four state equation the matrices $J_i$ become large. In this case, the Krylov subspace methods provide more efficient and accurate algorithms to compute matrix exponentials (see [8] for more details).

5. Numerical experiments

In this section, the performance of the linear filter introduced in this paper is illustrated by means of some numerical experiments. Specifically, the filter estimates $\hat{x}_{t_k/h}$ and $P_{t_k/h}$ are compared with the first two exact conditional moments of an autonomous linear state space model. The values $\hat{x}_{t_k/h}$ and $P_{t_k/h}$, for all $t_k$, were computed by Eqs. (5) and (6) using the values of $\hat{x}_{t_k/h}$ and $P_{t_k/h}$ computed by expressions (23) and (24).

Consider the linear model with state equation

$$\begin{align*}
\dot{x} &= \begin{bmatrix} -5 & 1 \\ 2 & -6 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} dt \\
&+ \begin{bmatrix} -0.25 & 0 \\ 0.25 & 0 \end{bmatrix} x + \begin{bmatrix} 0.5 \\ 0.25 \end{bmatrix} dw_1 \\
&+ \begin{bmatrix} 0 & 0.5 \\ 0 & -0.75 \end{bmatrix} x + \begin{bmatrix} 0.5 \\ 0.25 \end{bmatrix} dw_2
\end{align*}$$

(25)

and observation equation

$$z_k = x_1(t_k) + x_1(t_k)\xi_k + e_k,$$

(26)

where $\{\xi_k; \xi_k \sim \mathcal{N}(0, \lambda^2)\}$ and $\{e_k; e_k \sim \mathcal{N}(0, \sigma^2)\}$ are sequences of random vectors i.i.d, and $\phi(\xi_k, e_k) = 0$.

Fig. 1 shows a realization of the solution of the state equation (25) computed by the explicit Euler–Maruyama scheme [13] at the instant of time $t_j = j\Delta$, with $\Delta = 0.001$ and $j = 0, \ldots, 4000$.

Fig. 2 shows a realization of the observed Eq. (26) with $\lambda^2 = 0$ and $\sigma^2 = 100$. Figs. 3 and 4 present, respectively, the first and second exact conditional moments together the estimated $\hat{x}_{t_k/h}$ and $P_{t_k/h}$. $\hat{x}_{t_k/h}$ and $P_{t_k/h}$ were computed from the set of observations $\{z_k\}_{k=1,\ldots,N}$, with $t_k = kh$, $h = 0.01$ and $N = 400$.

In Fig. 5, a realization of the observed Eq. (26) with $\lambda^2 = 600$ and $\sigma^2 = 0$ is displayed. Figs. 6 and 7 present, respectively, the first and second exact conditional moments together the estimated $\hat{x}_{t_k/h}$ and $P_{t_k/h}$.

Fig. 8 shows a realization of the observed Eq. (26) with $\lambda^2 = 600$ and $\sigma^2 = 100$. Figs. 9 and 10 display, respectively, the first and second exact conditional moments together estimated $\hat{x}_{t_k/h}$ and $P_{t_k/h}$.

Note that, in all the cases, there is no significative difference between the exact and the estimated values of the first two conditional moments, which illustrate the effectiveness of the linear filter.

The simulation programs were developed in Matlab code (version 5.3.1) on an PC (Pentium III-1 GHz). For the above example, the computation of the filter estimates at each instant of time $t_k$ took 6 mseg. This result suggests that, for many practical problems, the algorithms introduced in the above section could be used for online filter implementations. Note also that, the performance of these algorithms could be improved by using a faster computer, a better computational code (for example C or Fortran) and an
appropriate method to compute matrix exponentials in the case of high dimensional filtering problems.

6. Conclusion

The optimal minimum variance linear filter for linear continuous-discrete models with multiplicative noise was constructed and their effectiveness was illustrated by means of numerical experiments.

At this point, it is opportune to point out that, this linear filter constitutes the kernel of the local linearization filters for nonlinear continuous-discrete models with multiplicative noise, which have been successful applied to solve several financial estimation problems [9].
Fig. 4. Second exact conditional moments together the estimated $P_{t_k|t_k}$, given the observation of Fig. 2. Top: $[P_{t_k|t_k}]_{11}$; Middle: $[P_{t_k|t_k}]_{12}$; Bottom: $[P_{t_k|t_k}]_{22}$.

Fig. 5. Realization of the observed Eq. (26) with $\lambda^2 = 600$ and $\sigma^2 = 0$.

Fig. 6. First exact conditional moment together estimated $\hat{x}_{t_k|t_k}$, given the observation of Fig. 5.
Fig. 7. Second exact conditional moments together estimated \( P_{h/h} \), given the observation of Fig. 5. Top: \([P_{h/h}]_{11}\); Middle: \([P_{h/h}]_{12}\); Bottom: \([P_{h/h}]_{22}\).

Fig. 8. Realization of the observed Eq. (26) with \( \lambda^2 = 600 \) and \( \sigma^2 = 100 \).

Fig. 9. First exact conditional moment together the estimated \( \hat{x}_{h/h} \), given the observation of Fig. 8.
Fig. 10. Second exact conditional moments together the estimated $P_{k/q}$, given the observation of Fig. 8. Top: $[P_{k/q}]_{11}$; Middle: $[P_{k/q}]_{12}$; Bottom: $[P_{k/q}]_{22}$.

Acknowledgements

The authors are very grateful to Prof. V.S. Kouikoglou from the Technical University of Create for providing them current information related with the content of this paper and, also, to the Ministry of Education, Science and Culture of Japan and to the Mizuho Trust & Banking Co. Ltd. for the partial support of this paper.

References


