A statistical method of estimation and simulation for systems of stochastic differential equations

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SUMMARY
This paper proposes a new statistical method of estimation and simulation for systems of stochastic differential equations. By using this method, we can easily derive a discretised process from a system of stochastic differential equations. The discretised process is tractable enough to simulate the system and to construct the likelihood which is used to estimate the coefficients of the stochastic differential equations from discrete observation.

Some key words: Discretisation; Local linearisation method; Maximum likelihood estimation; Systems of stochastic differential equations.

1. Introduction
In order to approximate solutions of stochastic differential equations and to estimate their coefficients from discrete observation, various approaches have been investigated. For example, numerical approximation of stochastic differential equations are extensively studied in Milstein (1974, 1978), Rümelin (1983), Kloeden & Platen (1992) and their references. In estimation, Gallant & Long (1997) discuss the minimum chi-squared estimator and its efficiency. While these methods are useful for simulation and estimation, it may be more convenient to have a method which can be used for both purposes. The aim of this paper is to propose a method which can be used for estimation and simulation for stochastic differential equations.

We have already developed such a method, called a local linearisation method, applicable to one-dimensional cases in Shoji (1998), where the rate of convergence of the method is discussed. In Shoji & Ozaki (1997) the proposed method shows good performance in estimation in a numerical comparison with alternative methods. Thus, we concentrate on the local linearisation method. The basic idea of the method is that a given nonlinear stochastic differential equation is locally approximated by a linear stochastic differential equation. This approximation technique contrasts with the techniques known as Runge–Kutta methods studied in Rümelin (1983) and Kloeden & Platen (1992) in that the former approximates an original stochastic differential equation by a linear stochastic differential equation and the latter does it by a stochastic difference equation.

Although one-dimensional stochastic differential equations are tractable from theoretical and numerical viewpoints, because of their simplicity their descriptive power is not strong enough to explain fully the stochastic nature of real dynamics that generally exhibit the influence of various contributions. Thus, the natural extension is to use systems of stochastic differential equations
2. Discretisation

We set up the following system of stochastic differential equations:

\[ dx_t = f(x_t) \, dt + \sigma \, dB_t, \]  

where \( x_t \) is an \( n \)-dimensional stochastic process, \( f(x_t) \) is an \( n \)-valued smooth function which satisfies the Lipschitz condition and whose Jacobian is not zero, \( \sigma \) is an \( n \times d \) constant matrix with full rank, and \( B_t \) is a \( d \)-dimensional standard Brownian motion. The local linearisation method locally linearises the drift function \( f \). Naturally the local behaviour of \( f \) is expressed by the differential of \( f \). By Ito's formula,

\[ df = J \, dx + M \, dt, \]

where

\[
J = \frac{\partial (f_1, \ldots, f_n)}{\partial (x_1, \ldots, x_n)}, \quad M = \left\{ \frac{1}{2} \text{tr}(\sigma' H_1), \ldots, \frac{1}{2} \text{tr}(\sigma' H_n) \right\}, \quad H_i = \left( \frac{\partial^2 f_i}{\partial x_k \partial x_l} \right)_{1 \leq k, l \leq n}.
\]

Here, we assume that \( J \) and \( M \) are constant over \( t \in [s, s + \Delta t] \); that is, \( J_t = J \) and \( M_t = M_s \) for \( t \in [s, s + \Delta t] \). Using this assumption, linear approximation of \( f \) is as follows:

\[ f(x_t) \approx J_s x_t + M_s t + f(x_s) - J_s x_s - M_s. \]

If we replace \( f \) of (1) by (3), the original stochastic differential equation (1) is approximated by the following linear stochastic differential equation:

\[ dx_t = \{ J_s x_t + M_s t + f(x_s) - J_s x_s - M_s t \} \, dt + \sigma \, dB_t. \]

The linear stochastic differential equation can be solved explicitly and its discretised process is obtained as a solution to it. If we transform \( x_t \) into \( y_t = \exp(-J_t x_t) \), where

\[ \exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}, \]

\( y_t \) satisfies the stochastic differential equation:

\[ dy_t = \exp(-J_s t) M_s t \, dt + \exp(-J_s t) \{ f(x_s) - J_s x_s - M_s t \} \, dt + \exp(-J_s t) \sigma \, dB_t. \]

If we integrate both sides of (5), we obtain

\[ y_t - y_s = \{ \exp(-J_s t) - \exp(-J_s s) \} x_s + \frac{J_s^{-1} \{ \exp(-J_s s) - \exp(-J_s t) \} f(x_s)}{2} + \int_s^t \exp(-J_s u) \sigma \, dB_u. \]

Transforming \( y \) back into \( x \), we obtain finally the discretised process

\[
x_t = x_s + \frac{J_s^{-1} \{ \exp \{ J_s (t - s) \} - I \} f(x_s) + (J_s^{-1})^2 \{ \exp \{ J_s (t - s) \} - I - J_s (t - s) \} M_s}{2} + \int_s^t \exp \{ J_s (t - u) \} \sigma \, dB_u,
\]

where \( I \) is the \( n \times n \) identity matrix.

The stochastic integral in (6) follows the normal distribution with mean zero and covariance matrix

\[ V_s = \int_s^t \exp \{ J_s (t - u) \} \sigma \sigma' \exp \{ J_s (t - u) \} \, du; \]
see, for example, Karatzas & Shreve (1991, p. 364). In view of the linearity of the solution process, its transition probability density function is
\[
p(x_t | x_s) = \left(\frac{(2\pi)^n}{|V_s|}\right)^{-\frac{1}{2}} \exp \left\{-\frac{1}{2}(x_t - x_{ts})'V_s^{-1}(x_t - x_{ts})\right\},
\]
where
\[
x_{ts} = x_s + J_s^{-1}[\exp \{J_s(t - s)\} - I]f(x_s) + (J_s^{-1})^2[\exp \{J_s(t - s)\} - I - J_s(t - s)]M_s.
\]

At first sight, it seems difficult to calculate \(V_s\) in practice. However, \(V_s\) is obtained as a solution to a linear matrix equation. Applying integration by parts to (7), we get
\[
J_sV_s + V_sJ_s' = \exp \{J_s(t - s)\} \sigma \exp \{J_s'(t - s)\} - \sigma'.'
\]

If \(J_s\) is known, \(V_s\) is a solution to the above linear matrix equation. The following proposition guarantees the existence and uniqueness of the solution.

**Proposition 1.** Suppose that \(J_s\) in (9) has no pair of reverse-sign eigenvalues; that is, for each eigenvalue of \(J_s\), \(\lambda\), say, \(-\lambda\) is not an eigenvalue of \(J_s\). Then there is a unique solution of the matrix equation (9).

**Proof.** First, it is clear that there exists a solution since \(V_s\) is given by the integral form defined in (7). Next, we prove the uniqueness of a solution. Suppose that there exists another solution, \(V_0\), say. Then
\[
J_sV_s + V_sJ_s' = W, \quad J_sV_0 + V_0J_s' = W,
\]
where \(W = \exp \{J_s(t - s)\} \sigma \exp \{J_s'(t - s)\} - \sigma'\). By subtracting the second equation from the first equation we get
\[
J_s(V_s - V_0) = (V_s - V_0)(-J_s').
\]

In general, the linear matrix equation \(AX = XB\) has no solution other than the zero vector if \(A\) and \(B\) have no common eigenvalue; see, for example, Sugiura (1976, p. 239). Using the assumption and the fact that all the eigenvalues of \(J_s\) are the same as those of \(J_s'\), we see that \(J_s\) and \(-J_s'\) have no common eigenvalue. Note that \(J_s\) has no zero eigenvalue because of the assumption of the existence of \(J_s^{-1}\). Therefore, the above linear matrix equation with respect to \(V_s - V_0\) has no solution other than the zero vector. This shows the uniqueness of a solution. \(\square\)

Although the assumption of the proposition seems somewhat restrictive, in practice we rarely encounter a case in which \(J_s\) has reverse-sign eigenvalues. This is similar to the computation of eigenvalues and eigenvectors. For example, conventional algorithms for the computation exclude multiple roots.

Without the proposition, we may obtain a simple form of \(V_s\) when \((J_sV_s)' = J_sV_s\). In this case, the left-hand side of (9) is \(2J_sV_s\). Thus,
\[
V_s = \frac{1}{2}J_s^{-1}[\exp \{J_s(t - s)\} \sigma \exp \{J_s'(t - s)\} - \sigma'\].
\]

In particular, in the case of one-dimensional stochastic differential equations, \(V_s\) clearly satisfies \((J_sV_s)' = J_sV_s\).

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**References**


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