THE STATISTICAL ANALYSIS OF PERTURBED LIMIT CYCLE PROCESSES USING NONLINEAR TIME SERIES MODELS

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Abstract. Statistical analysis of perturbed limit cycle process is discussed. Limit cycles of van der Pol equation type are considered and statistical time series models for the perturbed limit cycle processes are presented. It is shown that the model shares some interesting qualitative properties although the presented models are different, in appearance, from the van der Pol equation. Applications of the model to the statistical analysis of Canadian Lynx data are also discussed with numerical results.

Keywords. Non-linear time series model; Non-linear random vibrations; Autoregressive model; Limit cycle; Singular point; Stability; Stationarity; van der Pol equation; Asymmetry; Spectrum; Higher harmonics.

1. INTRODUCTION

Recent developments in the theory and application of linear time series models have provided us with very powerful and convenient tools for the forecasting and control of general time series. However, as was pointed out by many statisticians in the Discussion of the Papers by Campbell et al. (1977) and Tong (1977), such models cannot explain typical nonlinear phenomena of time series in the real world such as self-sustained stochastic cyclical behaviour, time irreversibility, asymmetry etc. The self-sustained type of oscillation is known to be realized in some deterministic nonlinear differential equations, e.g., the Lotka–Volterra equation which is a model of a predator–prey system in ecology and the van der Pol equation which describes the behaviour of a certain type of vacuum tube in electrical circuits (Hirsch and Smale, 1974). Although the stochastic version of the theory of nonlinear differential equations has not yet thoroughly evolved in the field of stochastic differential equations, in the field of time series analysis Ozaki (1980) introduced some nonlinear discrete time series models to explain typical nonlinear stochastic phenomena such as amplitude dependent frequency shifts and perturbed limit cycles (see also Haggan and Ozaki, 1980 and Ozaki and Oda, 1978).

In this paper the approach of Ozaki (1980) is extended and some models are given for stochastic self-sustained oscillations. Some geometric properties of the models are analysed and their applications to the statistical analysis of the Canadian Lynx data is discussed with numerical results.

2. LIMIT CYCLES AND TIME SERIES MODELS

The objective in this section is to provide time series models which show stochastic cyclic behaviour whose oscillations are sustained not by external
disturbances but by their own nonlinear structure. For this purpose, attention is focused on time series models for the oscillations by the van der Pol equation. Van der Pol type oscillations follow the following nonlinear differential equation,

\[ \ddot{x}(t) - b(\dot{x}(t) - \frac{1}{3}x(t)\dot{x}(t)^3) + \alpha x(t) = 0 \]  
(2.1)

or
\[ \ddot{x}(t) - b(1 - x(t)^2)x(t) + \alpha x(t) = 0. \]
(2.2)

For \(|x(t)| < 1\), the damping term of (2.2) is negative and hence \(|x(t)|\) tends to increase, while for \(|x(t)| > 1\), the damping term of (2.2) becomes positive, and so \(|x(t)|\) tends to decrease. The interplay of these two opposite effects produces a self-sustained type of oscillation, which is given the name 'limit cycle' by Poincare. The limit cycle of the van der Pol equation is known to be orbitally stable, i.e., every trajectory approaches to this stable limit cycle as \(t \to \infty\) (Lienard, 1928). The damping term of nonlinear oscillations, in general, does not have a polynomial form as in (2.1) or (2.2), which implies the following nonlinear differential equation,

\[ \ddot{x}(t) + g(x(t))\dot{x}(t) + \alpha x(t) = 0 \]
(2.3)

where \(g(\cdot)\) is a nonlinear function. From the analogy of the van der Pol equation (2.2), an orbitally stable limit cycle is expected for (2.3) when \(g(x)\) is negative for small \(|x(t)|\) and positive for large \(|x(t)|\) (Levinson and Smith, 1942). Most oscillations in real world, however, are perturbed by external random disturbances \(n(t)\), and so perturbed limit cycle processes may be represented by the following nonlinear stochastic differential equation,

\[ \ddot{x}(t) + g(x(t))\dot{x}(t) + \alpha x(t) = n(t). \]
(2.4)

Statistical analysis of sampled time series of random vibrations has been done with linear time series models such as the autoregressive model of order \(p\) (AR \((p)\)),

\[ x_t = \phi_1 x_{t-1} + \cdots + \phi_p x_{t-p} + \epsilon_t \]
(2.5)
or the mixed autoregressive moving average model of order \(p\) and \(q\) (ARMA \((p, q)\)),

\[ x_t = \phi_1 x_{t-1} + \cdots + \phi_p x_{t-p} - \theta_1 \epsilon_{t-1} - \cdots - \theta_q \epsilon_{t-q} + \epsilon_t \]
(2.6)

Although the linear models are useful and convenient in real applications, they are not appropriate in revealing nonlinearities such as limit cycles. Ozaki (1980) introduced a nonlinear time series model,

\[ x_t = (\phi_1 + \pi_1 e^{-x_{t-1}^2/2})x_{t-1} + \cdots + (\phi_p + \pi_p e^{-x_{t-p}^2/2})x_{t-p} + \epsilon_t \]
(2.7)

for the statistical analysis of time series from general nonlinear random vibrations and explained limit cycles and other nonlinear phenomena using the model. The model was introduced using the idea of the parametric specification of the dynamics of the characteristic roots of an autoregressive model. This idea is a natural consequence of the understanding that the van der Pol equation (2.2)
or the well-known Duffing equation,
\[ x + cx + ax + \beta x^3 = F \cos \omega t \quad (2.8) \]
is derived from the parametric specification of the dynamics of the damping force \( g_1(x)\dot{x} \) or the restoring force \( g_2(x)x \) of the following general nonlinear differential equation,
\[ \ddot{x} + g_1(x)\dot{x} + g_2(x)x = F(t) \quad (2.9) \]
where \( F(t) \) is an external force. Note that the model (2.7) is introduced (Ozaki, 1980) for time series sampled from continuous random vibration data with a sufficiently small sampling interval. This means that the movement of the characteristic frequency is limited within a sufficiently low frequency area which is equivalent to saying that the arguments of the instantaneous characteristic roots of the model (2.7) stay within a sufficiently small range of values. If this assumption is removed, there may be many pathological situations which are not common in real vibration phenomena, although they are mathematically interesting.

It has been shown that the model (2.7) can exhibit some non-linear phenomena such as non-zero singular points and limit cycles observed in non-linear vibration theory, which is one of the main fields which have stimulated the development of non-linear differential equation theory. Although the notions of singular points and limit cycles of discrete time non-linear dynamical system seem obvious from the analogy of the continuous time case, it may be useful for the later discussions to give a definition of singular points and limit cycles of discrete time difference equation \( x_t = f(x_{t-1}, x_{t-2}, \ldots, x_{t-p}) \).

**Definition 1.** A singular point of \( x_t = f(x_{t-1}, x_{t-2}, \ldots, x_{t-p}) \) is defined as a point, which every trajectory of \( x_t = f(x_{t-1}, x_{t-2}, \ldots, x_{t-p}) \) beginning sufficiently near it approaches either for \( t \to \infty \) or for \( t \to -\infty \). If it approaches it for \( t \to \infty \) we call it stable singular point and if it approaches it for \( t \to -\infty \) we call it unstable singular point.

Obviously a singular point \( \xi \) satisfies \( \xi = f(\xi, \xi, \ldots, \xi) \).

**Definition 2.** A limit cycle of \( x_t = f(x_{t-1}, x_{t-2}, \ldots, x_{t-p}) \) is defined as an ‘isolated’ and ‘closed’ trajectory \( x_{t+1}, x_{t+2}, \ldots, x_{t+q} \) where \( q \) is a positive integer.

‘Closed’ means that if initial values \( (x_1, \ldots, x_p) \) belong to the limit cycle then \( (x_{1+kq}, \ldots, x_{p+kq}) = (x_1, \ldots, x_p) \) for any integer \( k \). ‘Isolated’ means that every trajectory beginning sufficiently near the limit cycle approaches it either for \( t \to \infty \) or \( t \to -\infty \). If it approaches it for \( t \to \infty \) we call it stable limit cycle and if it approaches it for \( t \to -\infty \) we call it unstable limit cycle. The definition of limit cycle by Tong and Lim (1980) is equivalent to our definition of stable limit cycles. We introduce the above generalized definition only to discuss the stability of limit cycles later. It may be reasonable to define the period of a limit cycle of a non-linear difference equation \( x_t = f(x_{t-1}, \ldots, x_{t-p}) \) by ‘the smallest positive integer \( q \)’ which satisfies the above conditions of definition 2. A singular point
can be regarded as a limit cycle of period 1, but we distinguish it from other limit cycles of period \( q \geq 2 \) because it has a significantly different physical meaning. In our definition non-integer periods are not allowed. However this does not mean a serious limitation of the models in applications, because if we are interested in getting more precise value of the frequency we can get it by making the sampling interval smaller.

Some conditions for the model (2.7) to have a limit cycle are discussed in Haggan and Ozaki (1981). However, the condition (iii) of Haggan and Ozaki (1981) is not a necessary condition for the model to have a limit cycle, because even if the model has a non-zero singular point the model may have a stable limit cycle when the non-zero singular point is unstable. It is easily shown that the stability condition for the non-zero singular points of the model (2.7) is given as follows.

**Condition A.** The absolute values of the characteristic roots of

\[
\Lambda^p - h_1 \Lambda^{p-1} - \cdots - h_p = 0
\]  

(2.10)

are all less than one, where

\[
h_1 = (\pi_1 + \phi_1 \Sigma \pi_j - \pi_1 \Sigma \phi_j) / \Sigma \pi_j + 2(1 - \Sigma \phi_j) \log \{(1 - \Sigma \phi_j) / \Sigma \pi_j\}
\]

\[
h_i = (\pi_i + \phi_i \Sigma \pi_j - \pi_i \Sigma \phi_j) / \Sigma \pi_j \quad (i = 2, 3, \ldots, p).
\]

The following model,

\[
x_t = (1.57 + 3 e^{-x^{2}_{t-1}}) x_{t-1} - (0.83 + 2 e^{-x^{2}_{t-1}}) x_{t-2} + \varepsilon_t
\]  

(2.11)

satisfies the conditions (i), (ii) and (iii) of Haggan and Ozaki (1981) and has a limit cycle when the white noise \( \varepsilon_t \) is suppressed. The model

\[
x_t = (1.8 + 4 e^{-x^{2}_{t-1}}) x_{t-1} - (0.97 + 0.1 e^{-x^{2}_{t-1}}) x_{t-2} + \varepsilon_t
\]  

(2.12)

satisfies the condition (i) and (ii) but not (iii) of Haggan and Ozaki (1981), and so it has non-zero singular points

\[
\xi = \pm 1.77 \ldots.
\]

However the model (2.12) does not satisfy the condition A, and so the non-zero singular points are unstable and it has a stable limit cycle.

In the same way as model (2.7) may give rise to unstable singular points, it is possible that unstable limit cycles may occur. When the model

\[
x_t = (\phi_1 + \pi_1 e^{-x^{2}_{t-1}}) x_{t-1} + \varepsilon_t
\]  

(2.13)

is known to have a limit cycle of period \( q \) a stability condition for the limit cycle is obtained as follows. Let the limit cycle of the model (2.13) be

\[
x_n, x_{n+1}, \ldots, x_{n+q-1}, x_{n+q} (= x_i),
\]

then a point \( x_t \) on a trajectory near the limit cycle is represented as

\[
x_t = x_i + \xi_t.
\]
Then it can easily be shown that
\[ \xi_t = \{\phi_1 + \pi_1(1-2x_{t-1}^2)e^{-x_{t-1}^2}\} \xi_{t-1} + o(\xi_{t-1}). \]

From this it can be seen that when the solution of the difference equation
\[ \xi_t = \{\phi_1 + \pi_1(1-2x_{t-1}^2)e^{-x_{t-1}^2}\} \xi_{t-1} \]
converges to zero as \( t \to \infty \) the limit cycle is orbitally stable. This result may be viewed as a discrete version of Poincaré's derivation of a differential equation with periodic coefficients for the stability check of limit cycles in nonlinear differential equations (see Minorsky, 1974, chapter 5). However it is not an easy task to solve a difference equation with periodic coefficients. What is required is to know whether \( \xi_t \) of (2.15) converges to zero or not, and this can be checked by seeing whether \( |\xi_{t+q}/\xi_t| \) is less than one or not. From the relation we get
\[
\begin{align*}
\xi_{t+q} &= \{\phi_1 + \pi_1(1-2x_{t+q-1}^2)e^{-x_{t+q-1}^2}\} \xi_{t+q-1} \\
&= \{\phi_1 + \pi_1(1-2x_{t+q-1}^2)e^{-x_{t+q-1}^2}\} \\
&\quad \times \{\phi_1 + \pi_1(1-2x_{t+q-2}^2)e^{-x_{t+q-2}^2}\} \xi_{t+q-2} \\
&= \{\phi_1 + \pi_1(1-2x_{t+q-1}^2)e^{-x_{t+q-1}^2}\} \{\phi_1 + \pi_1(1-2x_{t+q-2}^2)e^{-x_{t+q-2}^2}\} \\
&\quad \cdots \{\phi_1 + \pi_1(1-2x_{t}^2)e^{-x_{t}^2}\} \xi_t.
\end{align*}
\]
This gives rise to the following stability theorem for limit cycles.

**Theorem 1.** A limit cycle of period \( q, x_{t+1}, x_{t+2}, \ldots, x_{t+q} \), of the model (2.13) is orbitally stable if
\[
|\{\phi_1 + \pi_1(1-2x_{t+q-1}^2)e^{-x_{t+q-1}^2}\} \\
\times \{\phi_1 + \pi_1(1-2x_{t+q-2}^2)e^{-x_{t+q-2}^2}\} \cdots \{\phi_1 + \pi_1(1-2x_{t}^2)e^{-x_{t}^2}\}| < 1. \tag{2.16}
\]

Using an analogous method the following stability theorem may easily be obtained for a limit cycle of a general order model (2.7).

**Theorem 2.** A limit cycle of period \( q, x_{t+1}, \ldots, x_{t+q} \) of the model (2.7) is orbitally stable when all the eigenvalues of the matrix,
\[
A = A_q \cdot A_{q-1} \cdots A_1
\]
have absolute value less than one, where
\[
A_t = \begin{pmatrix}
\phi_1 + \left[ \pi_1 - \sum_{i=1}^{q} (\pi x_{t+i-1})x_{t+i-1} \right] e^{-x_{t+i-1}^2} & \phi_2 + \pi_2 e^{-x_{t+i-1}^2} & \cdots & \phi_p + \pi_p e^{-x_{t+i-1}^2} \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0
\end{pmatrix}
\]
The model (2.7) was extended by Ozaki (1979a) to
\[ x_t = (\phi_1 + f_1(x_{t-1})e^{-x_{t-1}^2})x_{t-1} + \cdots + (\phi_p + f_p(x_{t-1})e^{-x_{t-1}^2})x_{t-p} + \epsilon_t \] (2.17)
with the object of giving a more sophisticated specification of the dynamics of the characteristic roots of AR model using Hermite type polynomials \( f_i(x_{t-1})e^{-x_{t-1}^2} (i = 1, \ldots, p) \), where
\[ f_i(x_{t-1}) = \pi_0^{(i)} + \pi_1^{(i)} x_{t-1} + \cdots + \pi_n^{(i)} x_{t-1}^n \quad (i = 1, \ldots, p). \]
Some interesting structural properties of the extended model are given in Ozaki (1979a).

We note that a different type of non-linear autoregressive model is obtained by employing a different type of approximation of dynamics of the instantaneous characteristic roots (see Ozaki, 1981). We also note that a general non-parametric approach to the specification of dynamics of autoregressive coefficients is discussed in Priestley (1981).

When the nonlinear vibrations follow the nonlinear stochastic differential equation excited by Gaussian white noise,
\[ \ddot{x}(t) + g_1(x(t))\dot{x}(t) + g_2(x(t))x(t) = n(t), \] (2.18)
this could be closely approximated by the second order nonlinear AR model (2.17), i.e., \( p = 2 \), where \( g_1(\cdot) \) and \( g_2(\cdot) \) are nonlinear functions. However it is often the case that a nonlinear vibrating system is excited by coloured Gaussian noise \( y(t) \) so that
\[ \ddot{x}(t) + g_1(x(t))\dot{x}(t) + g_2(x(t))x(t) = y(t). \] (2.19)
Since any coloured Gaussian noise may be closely approximated by a Gaussian autoregressive model of sufficiently high order, the following time series model (see fig. 1) is obtained
\[ x_t = (\phi_1 + (\pi_0^{(1)} + \pi_1^{(1)} x_{t-1} + \cdots + \pi_n^{(1)} x_{t-1}^n)e^{-x_{t-1}^2})x_{t-1} + (\phi_2 + (\pi_0^{(2)} + \pi_1^{(2)} x_{t-1} + \cdots + \pi_n^{(2)} x_{t-1}^n)e^{-x_{t-1}^2})x_{t-2} + y_t \] (2.20)
where \( y_t = \theta_1 y_{t-1} + \cdots + \theta_p y_{t-p} + \epsilon_t \) and
\[ \epsilon_t \sim N(0, \sigma^2_\epsilon). \]

![Diagram](image)

**Figure 1.**

There is also the case when the nonlinear vibrating system (2.18) is observed through a coloured filter, for example
\[ \ddot{x}(t) + c_1 \ddot{x}(t) + c_2 \dot{x}(t) + c_3 x(t) = y(t) \] (2.21)
where
\[ \ddot{y}(t) + g_1(y(t))\dot{y}(t) + g_2(y(t))y(t) = n(t). \]

Time series data sampled from this kind of vibrating system may be represented by the following time series model (see fig. 2).
\[ x_t = \phi_1 x_{t-1} + \cdots + \phi_p x_{t-p} + y_t \tag{2.22} \]
where
\[ y_t = \{ \theta_1 + (\pi_1^{(1)} y_{t-1} + \cdots + \pi_{r_1}^{(1)} y_{t-r_1}) e^{-y_{t-1}^2} \} y_{t-1} + \{ \theta_2 + (\pi_1^{(2)} y_{t-1} + \cdots + \pi_{r_2}^{(2)} y_{t-r_2}) e^{-y_{t-1}^2} \} y_{t-2} + \varepsilon_t \]
and
\[ \varepsilon_t \sim N(0, \sigma_e^2). \]

\[ \varepsilon_t \]
\[ x_t \quad L_p \quad y_t \quad \log \]
\[ N_2 \quad \phi_1 \quad \phi_p \]

**Figure 2.**

It may easily be seen that the model (2.17) gives a stationary ergodic time series when the characteristic roots of
\[ \Lambda^p - \phi_1 \Lambda^{p-1} - \cdots - \phi_p = 0 \]
all lie inside the unit circle (Ozaki, 1979b). It is also easily seen that the stationarity and ergodicity conditions of the model (2.20) or (2.22) are that the characteristic roots of
\[ \Lambda^m - \phi_1 \Lambda^{m-1} - \cdots - \phi_m = 0 \tag{2.23} \]
and
\[ \Lambda^n - \theta_1 \Lambda^{n-1} - \cdots - \theta_n = 0 \tag{2.24} \]
all lie inside the unit circle, where \( m = 2 \) and \( n = p \) for (2.20) and \( m = p, n = 2 \) for (2.22).

The statistical identification of the models is performed using ‘Entropy Maximization Principle’ of Akaike (1977). The entropy \( B(f; g) \) of the model \( g(z; x) \) for the true probability distribution \( f(z) \) is defined by
\[ B(f; g) = -\int \left\{ \frac{f(z)}{g(z; x)} \right\} \log \left( \frac{f(z)}{g(z; x)} \right) g(z; x) \, dz. \tag{2.25} \]
The so-called Akaike’s Information Criterion (AIC) defined by
\[ \text{AIC} = -2 \log \text{(maximum likelihood)} + 2 \text{(number of parameters)} \tag{2.26} \]
is a consistent estimate of \(-E_B(f; g)\) (see Akaike, 1973), and so the Minimum AIC Estimation (MAICE) procedure is a realization of the Entropy Maximization
Principle. In the MAICE procedure, then, parameters of the model are estimated using the Maximum Likelihood Method. Maximum likelihood estimates of the non-linear time series models (2.17), (2.20) and (2.22) are asymptotically equivalent to the least squares estimates when the series is ergodic because the initial effect dies out as time passes. Therefore the least squares estimation method given by Haggan and Ozaki (1981) is employed for the model.

\[ x_t = (\phi_1 + \pi_1 e^{-\gamma x_{t-1}})x_{t-1} + \cdots + (\phi_p + \pi_p e^{-\gamma x_{t-1}})x_{t-p} + \varepsilon_t \]  

(2.27)

where \( \gamma \) is a scaling parameter. For the model (2.20) or (2.22), however, the least squares estimates cannot be obtained by solving a linear equation, and a nonlinear optimization procedure is necessary in the computation of the least squares estimates.

3. CANADIAN LYNX DATA

In this section the models will be applied to the analysis of real data which may be regarded as a perturbed limit cycle process. Some well-known data which may be regarded in this way are the annual records of the number of Canadian Lynx trapped in the years 1821 to 1934 in the Mackenzie River district of North-West Canada. The Lynx data was first analysed by P. A. P. Moran (1953), and is known to be cyclic with period \( \hat{p} \) of about 10 years. Recently the data was re-analysed by many statisticians (Campbell and Walker, 1977; Tong, 1977; Tong and Lim, 1980; Haggan and Ozaki, 1981). Haggan and Ozaki (1981) considered the Lynx data as a van der Pol type perturbed limit cycle process, whose periodic components, amplitude and phase are also disturbed by random noise, and fitted a model of (2.27) type of eleventh order. The fitted model showed the presence of a limit cycle of period about 9.45 years when the white noise input is suppressed.

However there may be a criticism to Haggan and Ozaki’s model that the time series generated from the model appears almost symmetric, while the Lynx data is obviously not symmetric; the time spent in rising from a trough to a peak appears to be greater than the time spent in falling from a peak to a trough. The criticism may be obviated if a model of type (2.17) is employed where the orders \( r_1 \) and \( r_2 \) of the Hermite polynomials \( f_1(x_{t-1}) \) and \( f_2(x_{t-1}) \) are odd integers. (3.1) shows the estimated parameters of the model applied to the Lynx data where the autoregressive order is fixed to 2 and the order of polynomials of all the AR coefficients is fixed to be 1.

\[ x_t = \{0.138 + (0.316 + 0.982x_{t-1})e^{-\hat{\gamma}x_{t-1}}\}x_{t-1} \]

\[ -\{0.437 + (0.659 + 1.260x_{t-1})e^{-\hat{\gamma}x_{t-1}}\}x_{t-2} + \varepsilon_t \]  

(3.1)

\[ \hat{\gamma} = 3.89 \quad \hat{\delta}^2 = 0.4327 \times 10^{-1}. \]

In the limit cycle obtained by the model, more time is seen to take in rising from a trough to a peak than in falling from a peak to a trough: about five steps for rising and four or five steps for falling. However as is seen in the values of
residual variance and AIC of the model, the adequateness of fit of the model is not very good compared with the model of Haggan and Ozaki (1981).

Detailed analysis of the characteristic roots of Haggan and Ozaki's model shows that one pair of mutually conjugate complex roots which have the largest absolute value produces the cyclic behaviour with period about 10 years and other minor roots contribute to produce other minor peaks of high frequency in the spectrum (see fig. 1 of Tong (1977)), while in the model (3.1) these minor roots are ignored by fixing the autoregressive order to 2.

Taking account of these points, models of type (2.20) or type (2.22) can be considered for the Lynx data, where the order \( p \) of the linear part is taken to be 9 and the order of Hermite polynomials of the first and the second coefficients of nonlinear AR part is taken to be odd because of the asymmetry. When the order of Hermite polynomials of (2.22) is fixed to be 3, the following model is obtained,

\[
x_t = -0.481x_{t-1} - 0.247x_{t-2} + 0.318x_{t-3} + 0.230x_{t-4} + 0.352x_{t-5} \\
+ 0.096x_{t-6} - 0.085x_{t-7} - 0.289x_{t-8} - 0.181x_{t-9} + y_t \\
y_t = \{1.514 + (0.480 - 3.332y_{t-1} - 0.610y_{t-1}^2 + 8.906y_{t-1}^3)e^{-\gamma y_{t-1}}\}y_{t-1} \\
+ \{-0.902 + (-0.228 + 0.923y_{t-1} + 0.193y_{t-1}^2) \\
- 4.216y_{t-1}^3\}e^{-\gamma y_{t-1}}\}y_{t-2} + \varepsilon_t
\]

\[
\hat{\sigma}_x = 0.3153 \times 10^{-1} \quad \hat{\gamma} = 3.89
\]

The residual variance and the AIC value of the model shows that the model is much more adequate than the model (3.1). Which model (of (3.1) and (3.2)) should be taken will depend on the analyst's objectives. If he is concerned on the forecasting, the model (3.2) will be suitable although the model has many parameters, while if he is concerned only on the asymmetric limit cycle structure of the Lynx data, the model (3.1) will be sufficient.

Figure 3 shows the limit cycle of \( y \) of the above model (3.2) where the white noise \( \varepsilon_t \) is suppressed. The time spent in rising is 5 or 6 years and the time spent in falling is 4 or 5 years. Figure 4 shows the part of the simulation of \( y \), \((t = 1, \ldots, 1000)\), where the input is a Gaussian white noise \( \varepsilon_t \) with variance

![Figure 3.](image-url)
\( \sigma^2 = 0.3153 \times 10^{-1} \). Figure 5 shows the part of the simulation of \( x_t \) (\( t = 1, \ldots, 1000 \)) which is the output of the linear part of (3.1) where the above \( y_t \) is used as an input. Figure 6 shows the estimated spectrum of the above simulated \( y_t \) (\( t = 1, \ldots, 1000 \)) and fig. 7 shows the estimated spectrum of the above
simulated $x_t (t = 1, \ldots, 1000)$, which closely corresponds to the spectrum of the Lynx data (see fig. 1 of Tong (1977)).

However there is a view that these minor peaks in the spectrum are higher harmonics of the main frequency, caused by the nonlinear effect of the limit cycle. Actually, some van der Pol type models give us higher harmonics. For instance, the model

$$x_t = (1.95 + 0.23 e^{-x_t^{2.7}})x_{t-1} - (0.96 + 0.24 e^{-x_t^{2.7}})x_{t-2} + \epsilon_t \quad (3.3)$$

has a limit cycle (see fig. 13 of Haggan and Ozaki, 1980). Figure 8 shows the estimated spectrum of the simulation data of the model (3.3) where $\sigma_\epsilon^2 = 0.1 \times 10^{-2}$. In this figure higher harmonics are clearly observed. However, some of the limit cycle models such as the $y_t$ of the model (3.2) and some of the sampled data from analog simulation of perturbed van der Pol processes do not

**Figure 8.**

**Figure 9.**
give higher harmonics. Figure 9 shows the sampled analog simulation data of the perturbed van der Pol equation,

$$\ddot{x} - (\dot{x} - \frac{1}{3} x^3) + x = n$$

where $n$ is a Gaussian white noise. The estimated spectrum of the data does not show any higher harmonics (see fig. 10).

![Graph showing density in dB vs frequency](image)

**Figure 10.**

The study of the relation between the coefficients of the model and higher harmonics in a stochastic situation is a problem remaining for the future.

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