NON-LINEAR THRESHOLD AUTOREGRESSIVE MODELS FOR NON-LINEAR RANDOM VIBRATIONS

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Abstract

Time series models for non-linear random vibrations are discussed from the viewpoint of the specification of the dynamics of the damping and restoring force of vibrations, and a non-linear threshold autoregressive model is introduced. Typical non-linear phenomena of vibrations are demonstrated using the models. Stationarity conditions and some structural aspects of the model are briefly discussed. Applications of the model in the statistical analysis of real data are also shown with numerical results.

NON-LINEAR TIME SERIES MODEL; AUTOREGRESSIVE MODEL; THRESHOLD MODEL; NON-LINEAR VIBRATIONS; LIMIT CYCLE; AMPLITUDE-DEPENDENT FREQUENCY; DUFFING EQUATION; VAN DER POL EQUATION; SINGULAR POINT; STATIONARITY; JUMP PHENOMENA

1. Introduction

The analysis of random vibrations has been a considerable problem in various fields such as mechanical engineering, electrical engineering and acoustic engineering. In the statistical analysis of data sampled at discrete-time intervals from random vibrations in continuous time, linear time series models, such as AR or ARMA models, have been used (Gersch, Nielsen and Akaike (1973)). However, random vibrations are in general non-linear and a linear model is only an approximation to reality.

Well-known non-linear phenomena are jump phenomena which arise from Duffing’s equation,

\[ \ddot{x}(t) + c \dot{x}(t) + ax(t) + \beta x(t)^3 = F \sin \omega t; \]

the amplitude-dependent frequency shift which may be found in random vibrations, representable as a Duffing’s process, i.e.

\[ \ddot{x}(t) + c \dot{x}(t) + \alpha x(t) + \beta x(t)^3 = n(t), \]
where \( n(t) \) is a continuous-time white-noise process; and limit cycle behaviour which is typical of the van der Pol equation,

\[
\ddot{x}(t) - b(1 - x(t)^2)\dot{x}(t) + \alpha x(t) = 0.
\]

Ozaki (1980) introduced a non-linear amplitude-dependent autoregressive-type time series model and demonstrated that these non-linear phenomena may be represented by the model (see also Haggan and Ozaki (1980) and Ozaki and Oda (1977)).

On the other hand Tong (1978) introduced a non-linear time series model called the threshold AR model based on the so-called Sugawara tank model (Sugawara (1961)) for riverflow prediction in hydrology. Later Tong (1980) pointed out the usefulness of his model for non-linear random vibrations. The threshold AR model is usually composed of several linear AR models and the model is switched from one to another depending on the past values of the process. In this paper some limitations of linear threshold AR models for random vibrations are discussed and non-linear threshold AR models are introduced; these may be regarded as a non-linear extension of Tong (1980)'s linear threshold AR model, based on Ozaki (1980)'s idea of the specification of the dynamics of the damping and restoring forces of vibrations.

2. Random vibrations and time series models

Non-linear random vibrations represented by Equations (1.2) and (1.3) may be regarded as a special form of the general non-linear stochastic differential equation

\[
\ddot{x}(t) + g_1(x(t))\dot{x}(t) + g_2(x(t))x(t) = n(t)
\]

which may be written equivalently in state-space form

\[
\begin{pmatrix}
\dot{x}(t) \\
\dot{y}(t)
\end{pmatrix} =
\begin{pmatrix}
0 & 1 \\
-g_2(x(t)) & -g_1(x(t))
\end{pmatrix}
\begin{pmatrix}
x(t) \\
y(t)
\end{pmatrix} +
\begin{pmatrix}
0 \\
n(t)
\end{pmatrix}
\]

where \( g_1(\cdot) \) and \( g_2(\cdot) \) are non-linear functions, \( y = \dot{x}(t) \) and \( n(t) \) is a white-noise process. When the non-linear functions \( g_1(x) \) and \( g_2(x) \) are approximated by constants \( c \) and \( \alpha \) respectively, the linear stochastic differential equation,

\[
\ddot{x}(t) + cx(t) + \alpha x(t) = n(t)
\]

is obtained.

It is well known that the process defined by the linear stochastic differential equation (2.3) corresponds to the ARMA (2, 1) model,

\[
x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} - \theta_1 e_{t-1} + e_t,
\]

where \( e_t \) is a discrete-time white-noise process, provided the sampling interval is
taken to be sufficiently small. When the model (2.4) is quasi-periodic, the
damping of the system is characterized by the second coefficient \(\phi_2\), which is
minus the square of the absolute value of the characteristic root of the equation
\(\lambda^2 - \phi_2 \lambda - \phi_3 = 0\). If \(-\phi_2 > 1\) the system explodes, if \(-\phi_2 < 1\) the system dies
out, and if \(-\phi_2 = 1\) the system gives harmonic oscillations. On the other hand, in
the continuous model (2.3), the damping characteristic is specified by a damping
constant \(c\), where the system dies out if \(c > 0\), explodes if \(c < 0\) and gives
harmonic oscillations if \(c = 0\). The natural frequency \(\omega_0\) of the continuous system
(2.3) is given by \(\omega_0 = \sqrt{\alpha - c^2/4}/2\pi\), while in the discrete system (2.4), the
natural frequency \(f_0\) is given by \(f_0 = (1/2\pi)\tan^{-1}(\sqrt{-4\phi_2 - \phi_1^2/\phi_2})\). So, when the
damping coefficient \(\phi_2\) is fixed, the first coefficient, which is twice the real part of
the characteristic roots, characterizes the natural frequency of the system. If \(\phi_2\) is
increased, the frequency is decreased, and if \(\phi_1\) is decreased, then the frequency
is increased. For the hard-spring type of Duffing’s process (1.2) where \(\beta > 0\),
however, the natural frequency is known to be amplitude-dependent; the
frequency is increased when the amplitude \(|x(t)|\) is increased (Stoker (1950)).
For the case of discrete-time series, the hard-spring type of amplitude-dependent
frequency shift is realized (see Ozaki (1979a)) by modifying the first coefficient, for
instance, as follows:

\[
(2.5) \quad x_t = (1.5 + 0.28e^{-x_{t-1}^2})x_{t-1} - 0.96x_{t-2} + \varepsilon_t.
\]

The characteristic roots of this equation move between \(\mu_0 = 0.89 \pm 0.41i\) and
\(\mu = 0.75 \pm 0.63i\) when \(|x_{t-1}|\) changes between 0 and \(\infty\) (see Figure 1).

For the van der Pol equation (1.3), where \(b > 0\), the damping term \(-b(1-x^2)\)
becomes negative when \(|x| < 1\) and so the system begins to diverge. However,
for large \(|x| (x^2 > 1)\) the damping term becomes positive and the system begins
to die out. Therefore limit cyclic behaviour is observed for (1.3), and for the process
defined by the van der Pol-type stochastic differential equation,

\[
(2.6) \quad \ddot{x} - b(1-x^2)x + \alpha x = \varepsilon,
\]

perturbed limit cycle behaviour is expected to occur. In the discrete case, the
characteristic roots of the model

\[
(2.7) \quad x_t = (1.95 + 0.23e^{-x_{t-1}^2})x_{t-1} - (0.96 + 0.24e^{-x_{t-1}^2})x_{t-2} + \varepsilon_t
\]

move between \(\lambda_0 = 1.09 \pm 0.109i\) and \(\lambda = 0.975 \pm 0.0968i\) as \(|x_{t-1}|\) changes
between 0 and \(\infty\) (see Figure 1), and limit cycle behaviour actually does occur.

As is seen in the above examples, the Duffing-type process (1.2) or the van der
Pol-type process (2.6) may be introduced by specifying \(g_2(\cdot)\) or \(g_1(\cdot)\) of (2.1) by a
second-order polynomial, i.e. \(a + bx^2\) or \(c(1-x^2)\), while the model of Ozaki
(1980) is introduced by specifying \(h_1(\cdot)\) or \(h_2(\cdot)\) of the general model,

\[
(2.8) \quad x_t = h_1(x_{t-1})x_{t-1} + h_2(x_{t-1})x_{t-2} + \varepsilon_t
\]
as zeroth-order Hermite-type polynomials, $\phi_1 + \pi^{(1)} e^{-x_{t-1}^2}$ and $\phi_2 + \pi^{(2)} e^{-x_{t-1}^2}$. If more elaborate approximations of the dynamics of the characteristic roots of (2.8) are needed it is only necessary to use higher-order Hermite-type polynomials (Ozaki (1979c)).

On the other hand, Tong (1980) considered differential equations of non-linear vibrations in state-space form, $\dot{x}(t) = f(x(t))$. Tong (1980) says that time series modelling of non-linear vibrations is a modelling of the non-linear function $f(\cdot)$ in the discrete case,

$$\dot{x}_t = f(x_{t-1}),$$

(2.9)

and suggested the use of a piecewise linear approximation for $f(\cdot)$. Using this idea he introduced a general threshold AR model and demonstrated its use by means of interesting numerical applications (Tong (1980)). It is interesting to examine the threshold model for non-linear vibrations from the above-mentioned viewpoint of specifying the dynamics of the damping and restoring forces of general non-linear random vibrations.

As is shown in (2.2), Tong’s piecewise linear approximation of $f(\cdot)$ of (2.9) is regarded, in our formulation, as a step function approximation to $g(\cdot)$ of the form

$$x_t = g(x_{t-1})x_{t-1}.$$  

(2.10)

Actually, if instead of the model (2.7), we use a threshold model of the form

$$x_t = \begin{cases} 
1.95x_{t-1} - 0.96x_{t-2} + \varepsilon, & \text{if } |x_{t-1}| \geq 0.5 \\
2.18x_{t-1} - 1.20x_{t-2} + \varepsilon, & \text{if } |x_{t-1}| < 0.5
\end{cases}$$

(2.11)
the characteristic roots of the model jump from \( \lambda_0 \) to \( \lambda_w \) (or vice versa) depending on the amplitude (see Figures 1 and 2), while the path of the two roots \( \lambda_0 \) and \( \lambda_w \) is specified by a continuous Hermite-type polynomial in the model (2.7) (see Figure 2).

In Tong (1980)'s general model for non-linear vibrations, constant terms are included and play an essential role for van der Pol-type vibrations. However, the introduction of constant terms is not natural at least in vibrations, because if the vibration process \( x(t) \) starts from the zero initial state it stays at 0, i.e. the zero-state, \( x = 0 \) and \( \dot{x} = 0 \), is a singular point, stable for Duffing-type vibrations, and unstable for van der Pol-type vibrations.

The threshold AR model fitted for lynx data in Tong (1980) is composed of two AR models (see (4.3) of Tong (1980)). However, the characteristic roots of both AR models lie outside the unit circle. Therefore, if certain initial values are used the system may not have a limit cycle and may diverge. In fact, if the initial values are \( x_1 = 0 \), \( x_2 = -100 \) and \( x_3 = 0 \), the process diverges instead of having a limit cycle.

If the nature of the model's step function approximation to the dynamics of restoring or damping force is considered, together with the fact that the limit cycle of the van der Pol equation should be orbitally stable and independent of initial values except for the singular point \( x = 0 \) and \( \dot{x} = 0 \), it seems that threshold AR models cannot be expected to give a good approximation to non-linear random vibrations. A more appropriate formulation for non-linear random vibrations based on the threshold idea is given by the non-linear threshold AR model, described in the next section.

3. Non-linear threshold AR model

Instead of the step function approximation of \( g(\cdot) \) of (2.12), we can consider non-linear approximations. For instance, consider the model

![Figure 2](image-url)
\begin{equation}
    x_t = \begin{cases}
        1.95x_{t-1} - 0.96x_{t-2} + \varepsilon_t & \text{if } |x_{t-1}| \geq 1.0 \\
        (2.18 - 0.23x_{t-1}^2)x_{t-1} - (1.2 - 0.24x_{t-1}^2)x_{t-2} + \varepsilon_t & \text{if } |x_{t-1}| < 1.0
    \end{cases}
\end{equation}

whose characteristic roots move much more smoothly than the model (2.11) (see Figure 2). This idea leads to the following general model for non-linear random vibrations:

\begin{equation}
    x_t = \begin{cases}
        \phi_1 x_{t-1} + \cdots + \phi_p x_{t-p} + \varepsilon_t & \text{if } |x_{t-1}| \geq T \\
        f_1(x_{t-1})x_{t-1} + \cdots + f_p(x_{t-1})x_{t-p} + \varepsilon_t & \text{if } |x_{t-1}| < T
    \end{cases}
\end{equation}

where \( f_i(x) = \phi_i^{(i)} + \pi_i^{(i)}x + \cdots + \pi_i^{(i)}x^n \) and \( f_i(T) = \phi_i \) \( (i = 1, \cdots, p) \). To distinguish the model (3.2) from Tong's linear threshold AR model, the model is called the non-linear threshold AR model. This non-linear threshold model should naturally be expected to have interesting non-linear properties because of the more sophisticated nature of the approximation involved. The amplitude-dependent frequency shift phenomena and the jump phenomena of hard-spring-type Duffing process can be seen, for instance, in the model

\begin{equation}
    x_t = \begin{cases}
        1.0x_{t-1} - 0.96x_{t-2} + \varepsilon_t & \text{if } |x_{t-1}| \geq 1 \\
        (1.78 - 0.78x_{t-1}^2)x_{t-1} - 0.96x_{t-2} + \varepsilon_t & \text{if } |x_{t-1}| < 1
    \end{cases}
\end{equation}

whereas perturbed limit cyclic behaviour may be observed in the process defined by model (3.1).

As for the stationarity of the model, it is easily seen (see Ozaki (1979b)), using Tweedie's theorem (Tweedie (1975)), that the process defined by the model (3.2) is stationary and ergodic if the roots of \( \Lambda^p - \phi_1\Lambda^{p-1} - \cdots - \phi = 0 \) all lie inside the unit circle.

Concerning singular points, the linear threshold AR model has singular points only at the mean of each AR model, while the non-linear threshold AR model has wider structural aspects. For example, if the dynamics of the characteristic roots of the model are designed as is shown in Figure 3, the following stationary model, say, is obtained:

\begin{equation}
    x_t = \begin{cases}
        0.8x_{t-1} + \varepsilon_t & \text{for } |x_{t-1}| \geq 1.0 \\
        (0.8 + 1.3x_{t-1}^2 - 1.3x_{t-1}^4)x_{t-1} + \varepsilon_t & \text{for } |x_{t-1}| < 1.0
    \end{cases}
\end{equation}

It can be easily proved (see Ozaki (1979c)) that the model has two unstable singular points, \( \xi^*_1 = 0.4358 \ldots \) and \( \xi^*_1 = -0.4358 \ldots \), and three stable singular points, \( \xi_0 = 0 \), \( \xi^*_2 = 0.9 \) and \( \xi^*_2 = -0.9 \). Therefore a simulated process fluctuates around one of the three stable singular points and jumps from around one stable singular point to another depending on the white noise input (see Figure 4).
4. Discussions

The model was applied to the Canadian lynx data which has attracted the interest of many statisticians (Moran (1953), Campbell and Walker (1977), Tong (1977), (1980), Haggan and Ozaki (1981)). In common with the technique used by most other analysts, the logarithm to base 10 of the data is taken, and the sample mean is subtracted before the application of the model. Using an analogous identification procedure as in Haggan and Ozaki (1981), the following 11th-order non-linear threshold AR model is obtained:

\[
    x_t = (1.02 + 0.26X^2)x_{t-1} - (0.76 - 1.30X^2)x_{t-2} + (0.25 - 0.01X^2)x_{t-3} \\
    - (0.25 + 0.14X^2)x_{t-4} - (0.02 - 0.78X^2)x_{t-5} + (0.11 - 0.97X^2)x_{t-6} \\
    - (0.21 - 0.98X^2)x_{t-7} + (0.17 - 0.75X^2)x_{t-8} + (0.05 + 0.63X^2)x_{t-9} \\
    + (0.21 - 0.46X^2)x_{t-10} - (0.21 + 0.41X^2)x_{t-11} + \varepsilon_t,
\]
where \( X = \hat{T} \) if \( |x_{t-1}| \geq \hat{T} \) and \( X = x_{t-1} \) if \( |x_{t-1}| < \hat{T} \). The estimated threshold \( \hat{T} = 0.57 \), and the residual variance \( \hat{\sigma}^2 = 0.3180 \times 10^{-1} \) which is quite small compared with the residual variances of other linear models. The two mutually conjugate complex characteristic roots of the model move out of the unit circle when \( |x_{t-1}| \) approaches to 0, while if \( |x_{t-1}| \) becomes larger all the characteristic roots stay inside the unit circle, and so the estimated model is stationary. If the estimated model is simulated suppressing the white noise part a limit cycle of about 10 years period is obtained, as is shown in Figure 5, where the actual lynx data are used as initial values. Thus the identification procedure is easy to apply and actually works well. The only problem for this model in real application is that it may sometimes need too many parameters when the order of the model is large. To solve this problem, the same modelling method which is discussed in Ozaki (1979d) may be useful.

![Figure 5](image_url)

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**References**


Autoregressive models for random vibrations


